TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 209, 1975

ON THE REPRESENTATION OF LATTICES BY MODULES

BY

GEORGE HUTCHINSON

ABSTRACT. For a commutative ring R with unit, a lattice L is "representable by R-modules" if L is embeddable in the lattice of submodules of some unitary left R-module. A procedure is given for generating an infinite first-order axiomatization of the class of all lattices representable by R-modules. Each axiom is a universal Horn formula for lattices. The procedure for generating the axioms is closely related to the ring structure, and is "effective" in the sense that many nontrivial axioms can be obtained by moderate amounts of computation.

1. Introduction. Given a ring R with unit, let $\Gamma(M; R)$ denote the lattice of submodules of a unitary left R-module M. We may also regard $\Gamma(M; R)$ as the lattice of congruences of M [3, Theorem 1, p. 159]. These lattices are among the most important examples of modular lattices. Let $L(R)$ denote the class of all lattices representable by R-modules, that is, lattices L embeddable in $\Gamma(M(L); R)$ for some R-module $M(L)$. (All rings will have a unit and all modules will be unitary left modules hereafter.)

Several authors have considered $L(R)$ from the model theory viewpoint. The classes $L(R)$ are particular cases of a general class of models considered by B. M. Schein [18, Main Theorem, p. 15]. Schein's result implies that there exists a set of first-order universal lattice formulas axiomatizing $L(R)$ for any ring R, not necessarily commutative. Furthermore, it is known that a class of algebras is axiomatizable by universal Horn formulas if and only if it admits isomorphic images, subalgebras, products (including the one-element algebra) and ultraproducts [13], [14]. (For lattices as algebras, a universal Horn formula is the universal closure of either a lattice identity or a formula of the form:

$$[(e_1 = e_2) \,\&\, (e_3 = e_4) \,\&\, \cdots \,\&\, (e_{2n-3} = e_{2n-2})] \Rightarrow (e_{2n-1} = e_{2n}),$$

for lattice polynomials $e_1, e_2, \cdots, e_{2n}$ in some denumerable set of variables.) It is easily seen that $L(R)$ admits isomorphic images, sublattices and the trivial lattice, for any R. If L_i is embeddable in $\Gamma(M_i; R)$ for all $i \in I$, then $\Pi_{i\in I} L_i$ is embeddable in $\Gamma(\Pi_{i\in I} M_i; R)$ in an obvious way, and so $L(R)$ admits products. Since Schein's result implies that $L(R)$ admits ultraproducts, it follows that $L(R)$

Received by the editors April 10, 1972.
AMS (MOS) subject classifications (1970). Primary 06A30, 13C99.

is first-order axiomatizable by universal Horn formulas. C. Herrmann and W. Poguntke gave a general class of models admitting ultraproducts; their technique also shows that $L(R)$ admits ultraproducts for any R [6, Theorem 6, p. 5]. Also, Herrmann and Poguntke [6, Theorem 3, p. 2] show that $L(R)$ is not finitely first-order axiomatizable for various rings R, in particular, for the unitary subrings of the field $\mathbf{Q}$ of rationals. (They construct a lattice in $L(\mathbf{Q})$ which is an ultraproduct of certain lattices, none of which is representable by $\mathbf{Z}$-modules (abelian groups).) M. Makkai and G. McNulty [17] considered the method of Schein applied specifically to classes $L(R)$, and obtained some further results. In [17, Corollary 2], they use the method of W. Craig [4] to assert the existence of a primitive recursive set of universal Horn formulas axiomatizing $L(R)$ if R is defined on a recursive set and has recursive ring operations. (B. M. Schein's method of [18] can be used to prove that $L(R)$ has a recursive axiomatization by universal formulas if R is recursive, and he observes that the result still holds if R is a finitely-presented ring.) In [17, Theorem 3], it is shown that $L(R)$ depends only on the set of finite systems of equations satisfiable in R. By a finite system of equations is meant a set of formal ring equations, each of form $u_1 + u_2 = u_3$ or $u_1 u_2 = u_3$, where each term u_i is either a variable or 0 or 1. The system is satisfiable in R if some assignment of elements of R to the variables causes every equation to hold in R, with 0 and 1 interpreted as usual. In [6] and [17], there are also related axiomatization results for classes $L(K)$ of lattices L embeddable in some $\Gamma(M(L); R(L))$, where $R(L)$ belongs to a given first-order axiomatizable class K of rings and $M(L)$ is some $R(L)$-module.

Our main theorem, giving a Horn formula axiomatization of $L(R)$ for commutative R, requires a much longer proof than the model theory proofs. However, the result is more than an existence proof: individual axioms with desired properties can often be generated by reasonable amounts of computation. More important, the procedure generating the axiomatization of $L(R)$ is closely related to the ring structure of R, and so may yield further insights into the connection between rings R and lattices representable by R-modules.

Our approach adapts techniques from the classical coordinatization theory for projective geometries [1, Chapter 2] and for complemented modular lattices [20]. However, the representations are not obtained by direct constructions as in the classical theory, but indirectly via the embedding theorem for small abelian categories [15], [5]. In [7], a small abelian category $\mathbf{A}_L$ is constructed from any modular lattice L having a smallest element 0 and satisfying the property that, for each x in L, there exist y and z in L such that:

$$x \wedge y = x \wedge z = y \wedge z = 0, \qquad x \vee y = x \vee z = y \vee z.$$

(If this is so, we say that every element of L "can be tripled" and that L is an "abelian" lattice. In [7], a slightly more general definition of "abelian" lattice was given.) Using the embedding theorem for A_L, we obtain [7, Theorem 4.3, p. 182]: A lattice is representable by abelian groups if and only if it is embeddable in an interval sublattice of some abelian lattice. The construction of A_L resembles the regular ring construction in the proof of von Neumann's coordinatization theorem [20, Theorem 14.1, p. 208]: Every complemented modular lattice having a homogeneous basis of order $n \geqslant 4$ is isomorphic to the lattice of principal right ideals of some regular ring R. The sets $S(A, B)$ of [7, p. 163] used to construct abelian category morphisms are a "relativized" version of the sets L_{ij} [20, p. 95] used by von Neumann to construct the ring elements. Negative graphs of homomorphisms are used in both theories (see [20, pp. 133, 148] and [7, p. 157]). This leads to the same formula (von Staudt's multiplication) for ring multiplication in [20] and abelian category composition in [7]. Then similar associativity results are obtained (compare [20, Lemmas 5.1, 5.2, pp. 117–118] with [7, 3.5, 3.6, pp. 164–166]). The transitivity of perspectivity in [20, Theorem 2.2, p. 265] is closely related to the correspondence between projectivities and category isomorphisms [7, 3.27, p. 179]. However, the complementation axiom is not required in [7], so the theory is not restricted to regular rings. The "tripling" property used to define "abelian" lattice may be regarded as a strong form of the von Neumann requirement for a homogeneous basis. (Unfortunately, these connections were not pointed out to the author in time to cite von Neumann's work in [7].)

Many other authors have considered coordinatization problems since [20], but we will only mention the work of B. Jónsson related to the representation of modular lattices. In [10], Jónsson gives a lattice identity equivalent to Desargues' Theorem for a projective plane. This "Arguesian" identity is satisfied in every lattice isomorphic to a lattice of commuting equivalence relations on some set [10, Lemma 2.1, p. 196]. It is not satisfied in every modular lattice [10, Theorem 2.3, p. 198]. As a corollary, it follows that not every modular lattice is representable by abelian groups. (Note that a lattice representable by R-modules is representable by abelian groups and by commuting equivalence relations.) In [11, Theorem 2.14, pp. 308–309], Jónsson shows that a complemented modular lattice satisfying the Arguesian identity is representable by abelian groups. In the same paper [11, Theorem 3.6, p. 313], there is an example of an Arguesian (modular) lattice that is not representable by abelian groups. In [12, Theorem 2, p. 457], he gives an infinite family of universal lattice Horn formulas which are satisfied in a lattice if and only if it is representable by commuting equivalence relations.

The proof of our main theorem rests on a complex construction and lengthy computations verifying its properties. We consider this construction separately, in the third section. In the second section, the procedure for generating the axiomatization of $L(R)$ is described and motivated, the main theorem and related results are stated, and it is demonstrated that the main theorem follows from the asserted properties of the construction mentioned above.

This paper documents the main result announced in [8]. Other results announced in [8] and more recent results are given in [9]. In particular, we state a fundamental result [9, Theorem 1] below. Let R-Mod denote the abelian category of R-modules and R-linear maps. For any infinite cardinal m, let R-Mod(m) denote the exact full subcategory of R-Mod of all R-modules having cardinality less than m. Then for any rings with unit R and S, $L(R) \subset L(S)$ if and only if there exists an exact embedding functor R-Mod$(m) \to S$-Mod for every infinite cardinal number m.

It is an open question whether $L(R) \subset L(S)$ is equivalent to existence of an exact embedding functor R-Mod $\to$ S-Mod in all cases.

2. **The main theorem.** Let R be a commutative ring with unit. Our first goal is to define the procedure which generates an infinite axiomatization of $L(R)$. Essentially, we define certain configurations, called "R-frames", including a special one called the "initial" R-frame. Each R-frame determines a certain universal lattice Horn formula. Four operations are defined by which R-frames can be modified. A Horn formula is "R-constructible" if it is determined by an R-frame which can be obtained from the initial R-frame by a finite number of operations of these four types. The R-constructible Horn formulas are the axioms for $L(R)$. In the following, the passages labelled "procedure" contain the formal definitions, and the passages labelled "interpretation" explain the intended meanings of the definitions.

Procedure. Two denumerably infinite sets of variables are introduced: $B = \{b_0, b_1, b_2, \cdots\}$ and $X = \{x_\omega, x_0, x_1, x_2, \cdots\}$. Let $F_R(B)$ denote the free R-module with generating set B. Its elements are sums $\Sigma_{i \geq 0}\, r_i b_i$ for sequences $r_0, r_1, r_2, \cdots$ in R such that at most finitely many terms r_i are nonzero. Let $LP(X)$ denote the set of lattice polynomials generated by the set of variables X.

Interpretation. Let M be an R-module. The variables of B will correspond to elements of M, and the variables of X will correspond to submodules of M. The special variable x_ω will act like the trivial submodule 0 of M.

Procedure. An "R-frame" is an ordered triple $\langle \Psi, G, \alpha \rangle$ such that Ψ is a formula

$$(e_1 = e_2) \,\&\, (e_3 = e_4) \,\&\, \cdots \,\&\, (e_{2n-1} = e_{2n}),$$

for lattice polynomials $e_1, e_2, \cdots, e_{2n}$ in $LP(X)$, G is a finite (possibly empty) subset of $F_R(B)$, and α is a function $B_0 \to LP(X)$ for some finite B_0, $\{b_0\} \subset B_0 \subset B$, such that b_k is in B_0 if b_k has nonzero coefficient r_k in any $\Sigma_{i \geq 0}\, r_i b_i$ in G. Let Dom(α) denote the domain of α hereafter. The Horn formula "determined by"

$\langle\Psi, G, \alpha\rangle$ is the universal closure of the open formula:

$$\Psi \Rightarrow (x_0 = \alpha(b_0)).$$

The "initial" R-frame is $\langle\Psi_0, \varnothing, \alpha_0\rangle$, where Ψ_0 denotes the formula $x_0 = x_0$, $\mathrm{Dom}(\alpha_0) = \{b_0\}$ and $\alpha_0(b_0) = x_0$. (The formula determined by the initial R-frame is $(x_0)[(x_0 = x_0) \Rightarrow (x_0 = x_0)]$.)

Interpretation. Suppose f: $X \to \Gamma(M; R)$ is a function, assigning submodules of M to variables in X. There is a unique function $\bar{f}$: $LP(X) \to \Gamma(M; R)$ extending f and preserving meets and joins. Let $\langle\Psi, G, \alpha\rangle$ be an R-frame as described above, and say that f "satisfies" Ψ if $\bar{f}(e_{2k-1}) = \bar{f}(e_{2k})$ for every conjunct $e_{2k-1} = e_{2k}$ of Ψ. This is just the normal use of Ψ as the hypothesis of a Horn formula.

The terms G and α form a system of constraints, which determines a submodule of M if f is given. Let M^B denote the R-module of all functions $B \to M$, regarded as assignments of elements of M to variables of B. An element $g = \Sigma_{i\geq 0}\, r_i b_i$ of G is interpreted as the R-linear equation $g = 0$ in M. More precisely, the "R-linear set" g^* is the submodule of M^B given by

$$g^* = \left\{h \in M^B\colon \sum_{i\geq 0} r_i h(b_i) = 0\right\}.$$

Since only finitely many coefficients r_i are nonzero, the sum above is well defined. The "box" $f_*(\alpha)$ determined by α and f: $X \to \Gamma(M; R)$ is the submodule of M^B given by

$$f_*(\alpha) = \{h \in M^B\colon\ h(b_k) \in \bar{f}\alpha(b_k) \text{ if } b_k \in \mathrm{Dom}(\alpha)\}.$$

That is, α is regarded as formally specifying membership of b_k in $\alpha(b_k)$. Given f: $X \to \Gamma(M; R)$ and h: $B \to M$, α becomes a conjunction of statements about membership of elements of M in submodules of M.

Suppose $G = \{g_1, g_2, \cdots, g_n\}$. The "extended solution set" $\mu_0(G, \alpha, f)$ is the submodule of M^B given by

$$\mu_0(G, \alpha, f) = f_*(\alpha) \wedge g_1^* \wedge g_2^* \wedge \cdots \wedge g_n^*,$$

the intersection of a box and finitely many R-linear sets. Let π_0: $M^B \to M$ be the projection given by $\pi_0(h) = h(b_0)$. The "solution set" $\mu(G, \alpha, f)$ is $\pi_0[\mu_0(G, \alpha, f)]$, a submodule of M. That is, the "extended solutions" are the assignments of elements of M to variables in B satisfying the constraints G and α, and the "solutions" are elements of M which can be assigned to b_0 as part of an extended solution.

Procedure. Suppose $\langle\Psi, G, \alpha\rangle$ is an R-frame, b_k is in $\mathrm{Dom}(\alpha)$, and b_p and b_q are distinct variables not in $\mathrm{Dom}(\alpha)$. Then $\langle\Psi_1, G_1, \alpha_1\rangle$ is a "union augmentation" of $\langle\Psi, G, \alpha\rangle$ if

Ψ_1 is the conjunction Ψ & $(\alpha(b_k) \subset x_p \vee x_q)$,

$G_1 = G \cup \{b_k - b_p - b_q\}$, $\mathrm{Dom}(\alpha_1) = \mathrm{Dom}(\alpha) \cup \{b_p, b_q\}$,

$\alpha_1(b_p) = x_p$, $\alpha_1(b_q) = x_q$ and $\alpha_1(b_i) = \alpha(b_i)$ for b_i in $\mathrm{Dom}(\alpha)$.

A union augmentation of an R-frame is an R-frame.

Interpretation. Union augmentation abstracts the principle that join equals sum in $\Gamma(M; R)$. ("If $b_k \in x_p \vee x_q$, then there exist $b_p \in x_p$ and $b_q \in x_q$ such that $b_k - b_p - b_q = 0$.") We need only the following:

2.1. *Suppose $\langle\Psi, G, \alpha\rangle$ is an R-frame and $\langle\Psi_1, G_1, \alpha_1\rangle$ is a union augmentation of it using $b_k - b_p - b_q$. Let f: $X \to \Gamma(M; R)$ satisfy Ψ_1, such that $\bar{f}(x_0) = \bar{f}\alpha(b_0) = \mu(G, \alpha, f)$. Then $\bar{f}(x_0) = \bar{f}\alpha_1(b_0) = \mu(G_1, \alpha_1, f)$.*

PROOF. Assume the hypotheses. Now p and q are nonzero, because b_p and b_q are not in $\mathrm{Dom}(\alpha)$, and so $\alpha(b_0) = \alpha_1(b_0)$ and $\mu(G_1, \alpha_1, f) \subset \pi_0[f_*(\alpha_1)] = \bar{f}\alpha_1(b_0) = \bar{f}(x_0)$. Let $v \in \bar{f}(x_0) = \mu(G, \alpha, f)$, so there exists h: $B \to M$ in $\mu_0(G, \alpha, f)$ such that $h(b_0) = v$. Since $h \in f_*(\alpha)$, $h(b_k) \in \bar{f}\alpha(b_k)$. Now $\alpha(b_k) \subset x_p \vee x_q$ is a conjunct of Ψ_1, so $\bar{f}\alpha(b_k) \subset \bar{f}(x_p) \vee \bar{f}(x_q)$. Choose $v_1 \in \bar{f}(x_p)$ and $v_2 \in \bar{f}(x_q)$ such that $h(b_k) = v_1 + v_2$. Let h_1: $B \to M$ be given by $h_1(b_p) = v_1$, $h_1(b_q) = v_2$ and $h_1(b_i) = h(b_i)$ for $i \neq p, q$. If $g = \Sigma_{j\geqslant 0}\, r_j b_j$ is in G, then $r_p = r_q = 0$ by the definition of R-frame, since b_p and b_q are not in $\mathrm{Dom}(\alpha)$. So, $\Sigma_{j\geqslant 0}\, r_j h_1(b_j) = \Sigma_{j\geqslant 0}\, r_j h(b_j) = 0$, and $h_1 \in g^*$. Using the assumptions on h, v_1 and v_2, we can show that $h_1 \in f_*(\alpha_1) \wedge (b_k - b_p - b_q)^*$. So, h_1 is in $\mu_0(G_1, \alpha_1, f)$, and $v = h_1(b_0)$ is in $\mu(G_1, \alpha_1, f)$. Therefore, $\bar{f}(x_0) \subset \mu(G_1, \alpha_1, f)$, completing the proof.

Procedure. Suppose $\langle\Psi, G, \alpha\rangle$ is an R-frame, b_k is not in $\mathrm{Dom}(\alpha)$, and $g = \Sigma_{i\geqslant 0}\, r_i b_i$ is in $F_R(B)$ such that $r_k = 1$ and $r_i = 0$ for all b_i not in $\mathrm{Dom}(\alpha) \cup \{b_k\}$. If $r_i \neq 0$ for some b_i in $\mathrm{Dom}(\alpha)$, let $\rho_0(g, b_k, \alpha)$ denote the join in $LP(X)$ of all terms $\alpha(b_i)$ such that $r_i \neq 0$ and $i \neq k$. Otherwise, let $\rho_0(g, b_k, \alpha)$ denote the variable x_ω. Then $\langle\Psi, G_2, \alpha_2\rangle$ is a "defined variable augmentation" of $\langle\Psi, G, \alpha\rangle$ if

$$G_2 = G \cup \{g\}, \qquad \mathrm{Dom}(\alpha_2) = \mathrm{Dom}(\alpha) \cup \{b_k\},$$

$$\alpha_2(b_k) = \rho_0(g, b_k, \alpha) \quad \text{and} \quad \alpha_2(b_i) = \alpha(b_i) \quad \text{for all } b_i \text{ in } \mathrm{Dom}(\alpha).$$

A defined variable augmentation of an R-frame is an R-frame.

Interpretation. If certain elements of an R-module belong to specified submodules, then any R-linear combination of the elements belongs to the join of the submodules. Defined variable augmentation and the "constraint reduction" operation discussed later both abstract the above principle.

2.2. *Suppose* $\langle \Psi, G, \alpha \rangle$ *is an R-frame and* $\langle \Psi, G_2, \alpha_2 \rangle$ *is a defined variable augmentation of it using* $g = \Sigma_{i \geq 0}\ r_i b_i$ *such that* b_k *is not in* Dom(α) *and* $r_k = 1$. *Let* $f\colon X \to \Gamma(M; R)$ *satisfy* Ψ, *and suppose that* $\bar{f}(x_0) = \bar{f}\alpha(b_0) = \mu(G, \alpha, f)$. *Then* $\bar{f}(x_0) = \bar{f}\alpha_2(b_0) = \mu(G_2, \alpha_2, f)$.

PROOF. Assume the hypotheses. Then $\mu(G_2, \alpha_2, f) \subset \bar{f}\alpha_2(b_0) = \bar{f}(x_0)$, as in 2.1. For $v \in \bar{f}(x_0)$, choose $h\colon B \to M$ in $\mu_0(G, \alpha, f)$ such that $h(b_0) = v$. Let $h_2\colon B \to M$ be given by $h_2(b_k) = h(b_k) - \Sigma_{j \geq 0}\ r_j h(b_j)$ and $h_2(b_i) = h(b_i)$ for $i \neq k$. As before, $h_2 \in g_0^*$ for $g_0 = \Sigma_{j \geq 0}\ s_j b_j$ in G, because b_k not in Dom(α) implies $s_k = 0$, and $h \in g_0^*$ was assumed. Also, $h_2 \in g^*$ because

$$\sum_{j \geq 0} r_j h_2(b_j) = \sum_{j=0}^{k-1} r_j h(b_j) + h_2(b_k) + \sum_{j>k} r_j h(b_j) = 0.$$

To prove $h_2 \in f_*(\alpha_2)$, consider two cases. If $g \neq b_k$, delete all terms with zero coefficient and let $g = b_k + \Sigma_{j=1}^{t}\ r_{i_j} b_{i_j}$ in $F_R(B)$ ($t \geq 1$, and $k, i_1, i_2, \cdots, i_t$ are distinct positive integers). Then $h_2 \in g^*$ implies

$$h_2(b_k) = -\sum_{j=1}^{t} r_{i_j} h_2(b_{i_j}) = -\sum_{j=1}^{t} r_{i_j} h(b_{i_j}).$$

Since $h(b_{i_j}) \in \bar{f}\alpha(b_{i_j})$, $h_2(b_k) \in \bigvee_{j=1}^{t} \bar{f}\alpha(b_{i_j}) = \bar{f}\rho_0(g, b_k, \alpha)$. So, $h_2 \in f_*(\alpha_2)$. If $g = b_k$, $h_2(b_k) = 0 \in \bar{f}(x_\omega) = \bar{f}\rho_0(g, b_k, \alpha)$, and again $h_2 \in f_*(\alpha_2)$. Therefore, $h_2 \in \mu_0(G_2, \alpha_2, f)$, and so $v \in \mu(G_2, \alpha_2, f)$. But then we have $\bar{f}(x_0) \subset \mu(G_2, \alpha_2, f)$, completing the proof.

Procedure. Suppose $\langle \Psi, G, \alpha \rangle$ is an R-frame, $G = \{g_1, g_2, \cdots, g_n\}$ and $g = \Sigma_{i=1}^{n}\ r_i g_i$ is in $F_R(B)$ for some $r_1, r_2, \cdots, r_n$ in R. Then $\langle \Psi, G_3, \alpha \rangle$ is called a "linear combination augmentation" of $\langle \Psi, G, \alpha \rangle$ if $G_3 = G \cup \{g\}$. A linear combination augmentation of an R-frame is an R-frame.

Interpretation. This operation abstracts the usual principle that a solution of a system of R-linear equations also satisfies any R-linear combination of system equations.

2.3. *Suppose* $\langle \Psi, G, \alpha \rangle$ *is an R-frame and* $\langle \Psi, G_3, \alpha \rangle$ *is a linear combination augmentation of it such that* $G_3 = G \cup \{g\}$. *Let* $f\colon X \to \Gamma(M; R)$ *satisfy* Ψ, *and suppose* $\bar{f}(x_0) = \bar{f}\alpha(b_0) = \mu(G, \alpha, f)$. *Then* $\bar{f}(x_0) = \bar{f}\alpha(b_0) = \mu(G_3, \alpha, f)$.

PROOF. Given g_1, g_2 in $F_R(B)$ and r in R, we can check that $(rg_1)^* \supset g_1^*$ and $(g_1 + g_2)^* \supset g_1^* \wedge g_2^*$. If $g = \Sigma_{i=1}^{n}\ r_i g_i$ for $G = \{g_1, g_2, \cdots, g_n\}$, then $g_1^* \wedge g_2^* \wedge \cdots \wedge g_n^* \subset g^*$. But then $\mu_0(G_3, \alpha, f) = \mu_0(G, \alpha, f)$, and the result follows.

Procedure. Suppose $\langle \Psi, G, \alpha \rangle$ is an R-frame and $g = \Sigma_{i \geq 0}\ r_i b_i$ is in G such that $r_k = 1$. Define $\rho_0(g, b_k, \alpha)$ in $LP(X)$ as before: If $g \neq b_k$, let $\rho_0(g, b_k, \alpha)$

denote the join in $LP(X)$ of all terms $\alpha(b_i)$ such that $i \neq k$ and $r_i \neq 0$. If $g = b_k$, let $\rho_0(g, b_k, \alpha)$ denote x_ω. Then $\langle \Psi, G, \alpha_4 \rangle$ is a "constraint decrease" of $\langle \Psi, G, \alpha \rangle$ if

$$\mathrm{Dom}(\alpha_4) = \mathrm{Dom}(\alpha), \quad \alpha_4(b_k) = \alpha(b_k) \wedge \rho_0(g, b_k, \alpha) \quad \text{and}$$

$$\alpha_4(b_i) = \alpha(b_i) \quad \text{for all } b_i \text{ in } \mathrm{Dom}(\alpha), i \neq k.$$

A constraint decrease of an R-frame is an R-frame.

Interpretation. Constraint decrease is the key operation, allowing modification of $\alpha(b_0)$.

2.4. *Suppose $\langle \Psi, G, \alpha \rangle$ is an R-frame and $\langle \Psi, G, \alpha_4 \rangle$ is a constraint decrease of it using $g = \Sigma_{i \geqslant 0} r_i b_i$ in G at b_k. So, $r_k = 1$. Let $f\colon X \to L$ satisfy Ψ, and $\bar{f}(x_0) = \bar{f}\alpha(b_0) = \mu(G, \alpha, f)$. Then $\bar{f}(x_0) = \bar{f}\alpha_4(b_0) = \mu(G, \alpha_4, f)$.*

Proof. Assume the hypotheses. Since $\alpha_4(b_0)$ is either $\alpha(b_0)$ or the meet of $\alpha(b_0)$ and some other lattice polynomial, we have $\mu(G, \alpha_4, f) \subset \bar{f}\alpha_4(b_0) \subset \bar{f}(x_0)$. For $v \in \bar{f}(x_0)$, choose $h\colon B \to M$ in $\mu_0(G, \alpha, f)$ such that $h(b_0) = v$ as usual. To prove that h is in $\mu_0(G, \alpha_4, f)$, it suffices to prove that $h(b_k) \in \bar{f}\alpha(b_k) \wedge \bar{f}\rho_0(g, b_k, \alpha)$. But $h(b_k) \in \bar{f}\alpha(b_k)$ is known, and $h \in g^*$ implies $h(b_k) \in \bar{f}\rho_0(g, b_k, \alpha)$ as in 2.2. This proves the proposition.

Procedure. A sequence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ $(n \geqslant 1)$ of R-frames is called "proper" if $\mathbf{u}_1$ is the initial R-frame and, for each i such that $1 \leqslant i < n$, $\mathbf{u}_{i+1}$ is either a union augmentation, a defined variable augmentation, a linear combination augmentation or a constraint decrease of $\mathbf{u}_i$. The formula determined by the last term $\mathbf{u}_n$ of a proper sequence is called "R-constructible". That is, if $\mathbf{u}_n = \langle \Psi, G, \alpha \rangle$, then the universal closure of $\Psi \Rightarrow (x_0 = \alpha(b_0))$ is an R-constructible universal Horn formula.

2.5. *If a lattice L is representable by R-modules, then every R-constructible Horn formula is satisfied in L.*

Proof. Let M be an R-module, and suppose that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ is a proper sequence of R-frames and $\mathbf{u}_n = \langle \Psi, G, \alpha \rangle$. An induction on n proves that $\bar{f}(x_0) = \bar{f}\alpha(b_0) = \mu(G, \alpha, f)$ if $f\colon X \to \Gamma(M; R)$ satisfies Ψ. (If $n = 1$, so $\mathbf{u}_n$ is the initial R-frame, this is easily proved. The induction step follows from 2.1, 2.2, 2.3 and 2.4.) So, the universal closure of the formula $\Psi \Rightarrow (x_0 = \alpha(b_0))$ is satisfied in $\Gamma(M; R)$. It then follows that every R-constructible universal Horn formula is satisfied in a lattice L that is representable by R-modules.

We have proved the less difficult half of our result, which can now be fully stated.

MAIN THEOREM. *Let R be a nontrivial commutative ring with unit. A lattice L is representable by R-modules if and only if every R-constructible universal Horn formula is satisfied in L.*

Two examples of Horn formula generation by our procedure are given below:

EXAMPLE 1. Consider the "Fano" lattice identity of R. Wille [21, p. 134]:

$$(x_1 \vee x_2) \wedge (x_3 \vee x_4) \subset [(x_1 \vee x_3) \wedge (x_2 \vee x_4)] \vee [(x_1 \vee x_4) \wedge (x_2 \vee x_3)].$$

If the above identity is satisfied in the projective geometry $\Gamma(M; F)$ of subspaces of a vector space M of dimension three or more over a division ring F, then the Fano postulate fails for every quadrangle in $\Gamma(M; F)$, and so $\mathrm{char}(F) = 2$ ([21], [2, pp. 37–38]). If R is a ring with unit, not necessarily commutative, and $\mathrm{char}(R) \neq 2$, then there exists a lattice in $L(R)$ for which the above identity is not satisfied. For example, let R^3 denote the free R-module generated by y_1, y_2 and y_3, and observe that the above identity fails if $x_1 = Ry_1$, $x_2 = R(y_1 - y_2)$, $x_3 = R(y_2 - y_3)$ and $x_4 = Ry_3$ in $\Gamma(R^3; R)$. We now show that the above identity is satisfied in every lattice in $L(R)$ if $\mathrm{char}(R) = 2$, for any ring R. Let $\mathbf{Z}_2$ denote the ring of integers modulo 2. Beginning with the initial $\mathbf{Z}_2$-frame, introduce two union augmentations using

$$g_1 = b_0 - b_1 - b_2 = b_0 + b_1 + b_2 \quad \text{and} \quad g_2 = b_0 - b_3 - b_4 = b_0 + b_3 + b_4.$$

Make defined variable augmentations at b_5 using $g_3 = b_1 + b_3 + b_5$ and at b_6 using $g_4 = b_1 + b_4 + b_6$. Then add $g_5 = g_1 + g_2 + g_3 = b_2 + b_4 + b_5, g_6 = g_1 + g_2 + g_4 = b_2 + b_3 + b_6$, and $g_7 = g_2 + g_3 + g_4 = b_0 + b_5 + b_6$ by linear combination augmentations. Finally, perform constraint decreases at b_5 using g_5, at b_6 using g_6 and at b_0 using g_7, in that order. The $\mathbf{Z}_2$-constructible Horn formula determined by the last term of this proper sequence is

$$(x_0, x_1, x_2, x_3, x_4)([(x_0 = x_0) \,\&\, (x_0 \subset x_1 \vee x_2) \,\&\, (x_0 \subset x_3 \vee x_4)]$$
$$\Rightarrow (x_0 \subset [(x_1 \vee x_3) \wedge (x_2 \vee x_4)] \vee [(x_1 \vee x_4) \wedge (x_2 \vee x_3)])).$$

This Horn formula is equivalent to the Fano identity above in any lattice. If $\mathrm{char}(R) = 2$, then $L(R) = L(\mathbf{Z}_2)$ [8], [9, Theorem 5(6)], and so the Fano identity is satisfied in every lattice in $L(R)$ by 2.5.

EXAMPLE 2. B. Jonsson's Arguesian lattice identity ([10], cited in [3, p. 109, #7]) is lattice equivalent to a Horn formula which is R-constructible for any commutative R. Beginning with the initial R-frame, make three union augmentations using $g_1 = b_0 - b_1 - b_4$, $g_2 = b_0 - b_2 - b_5$ and $g_3 = b_0 - b_3 - b_6$. Then make three defined variable augmentations using $g_4 = -b_1 + b_2 + b_7$, $g_5 = -b_1 + b_3 + b_8$ and $g_6 = -b_2 + b_3 + b_9$. Now introduce five linear combination augmentations, obtaining five new equations

$$g_7 = -g_1 + g_2 + g_4, \quad g_8 = -g_1 + g_3 + g_5, \quad g_9 = -g_2 + g_3 + g_6,$$

$$g_{10} = g_4 - g_5 + g_6 \text{ and } g_{11} = -g_4.$$

Perform constraint decreases at b_7 using g_7, at b_8 using g_8, at b_9 using g_9, at b_7 using g_{10}, at b_1 using g_{11}, at b_4 using g_7 and at b_0 using g_1, in that order. The Horn formula determined by the last term of this proper sequence is equivalent to the Arguesian identity in any lattice.

To complete the main theorem proof, we must use the complex construction mentioned previously. At this point, we will outline this construction $M(K; R)$ and describe its relevant properties. Consider first a generalization of [7, 4.2, 4.3, pp. 181–183]. Let M be an R-module, $\mathbf{N}$ the set of positive integers, and $M^{\mathbf{N}}$ the R-module of all functions $\mathbf{N} \to M$. Say that M_0 in $\Gamma(M^{\mathbf{N}}; R)$ has "finite support" if there exists n_0 in $\mathbf{N}$ such that $h \in M_0$ and $n > n_0$ imply $h(n) = 0$. That is, every element of M_0 has all coordinates zero outside a certain designated finite set of coordinates. Let $\Gamma_f(M^{\mathbf{N}}; R)$ denote the set of submodules of $M^{\mathbf{N}}$ having finite support. Just as in [7, 4.2], we can verify that $\Gamma_f(M^{\mathbf{N}}; R)$ is an ideal of $\Gamma(M^{\mathbf{N}}; R)$ and is an abelian lattice. Define a lattice embedding ψ_M: $\Gamma(M; R) \to \Gamma_f(M^{\mathbf{N}}; R)$ by

$$\psi_M(M_1) = \{h \in M^{\mathbf{N}}: h(1) \in M_1, h(n) = 0 \text{ for } n > 1\},$$

since $\psi_M(M_1)$ as defined above has finite support.

Let K be a (0, 1) lattice, that is, a lattice with a smallest element 0 and a largest element 1. We construct a lattice $M(K; R)$ abstracting $\Gamma_f(M^{\mathbf{N}}; R)$ together with a map ψ: $K \to M(K; R)$ which abstracts the embedding ψ_M: $\Gamma(M; R) \to \Gamma_f(M^{\mathbf{N}}; R)$. The points of $M(K; R)$ are equivalence classes of constraint systems. These new constraint systems (G, α) are slightly different from those appearing in R-frames $\langle \Psi, G, \alpha \rangle$. In particular, the new lattice constraint functions α have values in K rather than in $LP(X)$. Also, new variables are introduced so that the solution sets (given an embedding ι: $K \to \Gamma(M; R)$ such that $\iota(0) = 0$) will be elements of $\Gamma_f(M^{\mathbf{N}}; R)$ rather than submodules of M. The equivalence relation for the constraint systems is generated by seven rules, four of them resembling the R-frame operations. Under the given interpretation, equivalent constraint systems have the same solution set in $\Gamma_f(M^{\mathbf{N}}; R)$. We now assert the properties of $M(K; R)$ and ψ that will be established in the final section.

2.6. *For every* (0, 1) *lattice K, there exist an abelian lattice $M(K; R)$ and a lattice homomorphism ψ: $K \to M(K; R)$ such that ψ is an embedding if every R-constructible Horn formula is satisfied in K. Furthermore, for every object A of $\mathbf{A}_{M(K;R)}$, there exists a unit-preserving ring homomorphism ζ_A from R into* *the endomorphism ring* Hom(A, A) *in* $\mathrm{A}_{M(K;R)}$. *Finally, if* $f: A \to B$ *in* $\mathrm{A}_{M(K;R)}$ *and* $r \in R$, *then* $f\zeta_A(r) = \zeta_B(r)f$.

After two preparatory results, we can show that 2.6 suffices to prove the main theorem.

2.7. *If a lattice L satisfies a set* Σ *of universal lattice Horn formulas, then there exists a* (0, 1) *lattice K extending L such that every formula of* Σ *is satisfied in K.*

PROOF. Assume the hypotheses. Adjoin a smallest element to L if it does not already have one. Dually, adjoin a largest element if necessary. The resulting (0, 1) lattice K extends L. It can be shown that every finitely-generated sublattice of K can be embedded in L. But then every formula of Σ must be satisfied in K.

DEFINITION. If C is R-Mod or is a small abelian category, let $\Gamma(A; C)$ denote the lattice of subobjects of an object A of C.

2.8. *Let C be a small abelian category. Suppose that there exist ring homomorphisms* $\zeta_A: R \to \mathrm{Hom}_C(A, A)$ *preserving the ring unit, for every object A of C, and suppose that* $\zeta_B(r)f = f\zeta_A(r)$ *for every* $f: A \to B$ *in C and every r in R. Then there exists an exact embedding functor* $G: C \to R$-Mod.

PROOF. Assume the hypotheses. By the embedding theorem [15], [5], there exists an exact embedding functor $F: C \to$ Ab, where Ab is the category of abelian groups and homomorphisms. For each object A of C, $F(A)$ has an additive group structure. We make $F(A)$ into an R-module, denoted $G(A)$, by defining $rv = (F\zeta_A(r))(v)$ for r in R and v in $F(A)$. That is, $F\zeta_A(r) = r1_{G(A)}$. If $f: A \to B$ in C, then $Ff: G(A) \to G(B)$ is R-linear because $\zeta_B(r)f = f\zeta_A(r)$ holds for all r in R. So, $G(A)$ and $Gf = Ff$ determine a functor $G: C \to R$-Mod. Since F is an exact embedding, so is G.

OUTLINE OF MAIN THEOREM PROOF. Assume 2.6. Suppose that every R-constructible Horn formula is satisfied in L. By 2.7, there exists an embedding $L \to K$ for a (0, 1) lattice K such that every R-constructible Horn formula is satisfied in K. Let M denote $M(K; R)$; by 2.6 and 2.8 there exist a lattice embedding $\psi: K \to M$ and an exact embedding functor $G: \mathrm{A}_M \to R$-Mod. By restricting the codomain of ψ, we obtain a lattice embedding from K into the interval sublattice $M[\psi(0), \psi(1)]$ of M. If A denotes the object $\psi(1)/\psi(0)$ of A_M, then there exists a lattice isomorphism $M[\psi(0), \psi(1)] \to \Gamma(A; \mathrm{A}_M)$ by [7, 3.24, p. 178]. The exact embedding functor G induces an embedding $\Gamma(A; \mathrm{A}_M) \to \Gamma(G(A); R\text{-Mod})$; see [7, p. 183] for relevant information. So, L is representable by R-modules via the following composite of lattice embeddings:

$$L \to K \to M[\psi(0), \psi(1)] \to \Gamma(A; \mathrm{A}_M) \to \Gamma(G(A); R\text{-mod}).$$

(Of course, $\Gamma(G(A); R\text{-Mod})$ is isomorphic to $\Gamma(G(A); R)$.) This shows that 2.6 implies the reverse implication of the main theorem, and the forward implication was proved in 2.5.

Recall that every interval sublattice of an abelian lattice is representable by abelian groups [7, Theorem 4.3]. But then every **Z**-constructible Horn formula must be satisfied in any abelian lattice, and so all abelian lattices are representable by abelian groups. In fact, we can simplify [7, Theorem 4.3] as follows: A lattice is representable by abelian groups if and only if it is embeddable in some abelian lattice.

3. The construction of $M(K; R)$. Throughout this section, R will denote a fixed commutative ring with unit and K will denote a fixed (0, 1) lattice. To avoid confusion with other zeros, we will denote the smallest element of K by ω.

In the following, we will label certain explanatory material as "interpretation". Although this material is not needed for the formal definitions and calculations, it helps one to understand the ideas motivating the construction.

DEFINITION. Let V be $\{a_k, b_k: k = 1, 2, 3, \cdots\}$, a set of two denumerably infinite sequences of variables. We will sometimes write c_k to represent a general element a_k or b_k of V. Let $F_R(V)$ denote the free R-module with free generating set V. We will represent elements of $F_R(V)$ by coefficient functions. That is, h in $F_R(V)$ will be any function $h\colon V \to R$ such that $h(v) = 0$ except for at most finitely many v in V. A function $\alpha\colon V \to K$ is a "lattice constraint function" if $\alpha(v) = \omega$ except for at most finitely many v in V. A "constraint system" is a pair (G, α) such that G is a finite (possibly empty) subset of $F_R(V)$ and $\alpha\colon V \to K$ is a lattice constraint function. Let $D(K; R)$ denote the set of all constraint systems.

Interpretation. Suppose $\iota\colon K \to \Gamma(M; R)$ is an embedding for some R-module M, and $\iota(\omega) = 0$. We can modify the definitions of §2 to interpret $D(K; R)$. The variables a_k correspond to coordinate positions in $M^{\mathbf{N}}$, and the variables b_k $(k \geqslant 1)$ are similar to those in §2. The variable b_0 of §2 should now be identified with a_1. We obtain extended solutions in the R-module M^V similar to the extended solutions in M^B in §2, and project the extended solutions to obtain a solution set in $\Gamma_f(M^{\mathbf{N}}; R)$. More precisely, define the "R-linear set" g^* for g in $F_R(V)$ by

$$g^* = \left\{h \in M^V\colon \sum_{k \geqslant 1} (g(a_k)h(a_k) + g(b_k)h(b_k)) = 0\right\}.$$

Define the "box" $\iota_*(\alpha)$ for a lattice constraint function $\alpha\colon V \to K$ by

$$\iota_*(\alpha) = \{h \in M^V\colon h(c_k) \in \iota\alpha(c_k), \text{ all } c_k \in V\}.$$

Define the extended solution set $\nu_0(G, \alpha, \iota)$ to be the submodule

$$\iota_*(\alpha) \wedge g_1^* \wedge g_2^* \wedge \cdots \wedge g_n^*$$

of M^V, if $G = \{g_1, g_2, \cdots, g_n\}$. (If $G = \emptyset, \nu_0(G, \alpha, \iota) = \iota_*(\alpha)$.) Define the solution set to be the projection of the extended solution set. That is, let $\mathbf{a}$: $\mathbf{N} \rightarrow V$ be given by $\mathbf{a}(k) = a_k$, and let $\nu(G, \alpha, \iota)$ be the submodule $\{h\mathbf{a}: h \in \nu_0(G, \alpha, \iota)\}$ of $M^{\mathbf{N}}$. Because of the assumption $\iota(\omega) = 0$, $\nu(G, \alpha, \iota)$ is in $\Gamma_f(M^{\mathbf{N}}; R)$. The subsequent definitions and calculations are all motivated by this interpretation of $D(K; R)$.

DEFINITION. The equivalence relation $E(K; R)$ on $D(K; R)$ is defined by means of seven "rules of equivalence" E_1 through E_7. That is, binary relations E_1 through E_7 are defined on $D(K; R)$, and we say that $\mathbf{s}$ and $\mathbf{t}$ in $D(K; R)$ are "directly equivalent" if $\mathbf{s}E_i\mathbf{t}$ or $\mathbf{t}E_i\mathbf{s}$ for some i, $i \leqslant 7$. For $\mathbf{s}$ and $\mathbf{t}$ in $D(K; R)$, write $\mathbf{s} \sim \mathbf{t}$ if there exists a sequence $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_n$ $(n \geqslant 1)$ in $D(K; R)$ such that $\mathbf{s}_1 = \mathbf{s}$, $\mathbf{s}_n = \mathbf{t}$ and $\mathbf{s}_{i+1}$ is directly equivalent to $\mathbf{s}_i$ for $1 \leqslant i < n$. Then $E(K; R) = \{\langle \mathbf{s}, \mathbf{t} \rangle: \mathbf{s} \sim \mathbf{t}\}$ is an equivalence relation on $D(K; R)$.

For $G \subset F_R(V)$, say that v in V is "unused" in G if $g(v) = 0$ for all g in G.

If g is in $F_R(V)$, c_k is in V and α: $V \rightarrow K$, let $\rho(g, c_k, \alpha)$ denote the join in K of the finitely many elements $\alpha(v)$ such that $v \neq c_k$ and $g(v) \neq 0$. If $g(v) = 0$ for all $v \neq c_k$, let $\rho(g, c_k, \alpha)$ denote ω.

If $V_0 \subset V$ and g is in $F_R(V)$, let $\eta(g, V_0)$ denote the element h of $F_R(V)$ such that $h(v) = g(v)$ for v in V_0 and $h(v) = 0$ for v in $V - V_0$.

In the next paragraphs, assume that (G, α) and (H, β) are in $D(K; R)$.

Suppose c_k, b_p and b_q are distinct elements of V such that b_p and b_q are unused in G, and suppose that x_1 and x_2 are in K such that $\alpha(c_k) \subset x_1 \vee x_2$. Write $(G, \alpha) E_1 (H, \beta)$ if $H = G \cup \{c_k - b_p - b_q\}$, $\beta(b_p) = x_1$, $\beta(b_q) = x_2$ and $\beta(v) = \alpha(v)$ if $v \neq b_p, b_q$. Then (H, β) is called the "union augmentation of (G, α) using $c_k - b_p - b_q$ and x_1 and x_2 in K", and (G, α) is called a "union deletion of $c_k - b_p - b_q$ from (H, β)".

Suppose b_k is unused in G and g is in $F_R(V)$ such that $g(b_k) = 1$. Write (G, α) $E_2(H, \beta)$ if $H = G \cup \{g\}$, $\beta(b_k) = \rho(g, b_k, \alpha)$ and $\beta(v) = \alpha(v)$ if $v \neq b_k$. Then (H, β) is called a "defined variable augmentation of (G, α) using g at b_k", and (G, α) is called a "defined variable deletion of g from (H, β)".

Suppose $G = \{g_1, g_2, \cdots, g_n\}$ and g in $F_R(V)$ equals $\Sigma_{i=1}^n r_i g_i$ for some sequence $r_1, r_2, \cdots, r_n$ in R. Write $(G, \alpha) E_3 (H, \beta)$ if $H = G \cup \{g\}$ and $\beta = \alpha$. Then (H, β) is called the "linear combination augmentation of (G, α) by g", and (G, α) is called the "linear combination deletion of g from (H, β)".

Suppose there exist g in G and c_k in V such that $g(c_k) = 1$. Write $(G, \alpha) E_4$ (H, β) if $H = G$, $\beta(c_k) = \alpha(c_k) \wedge \rho(g, c_k, \alpha)$ and $\beta(v) = \alpha(v)$ for all $v \neq c_k$. Then

(H,β) is called the "constraint decrease of (G,α) at c_k using g", and (G,α) is called a "constraint increase of (H,β) at c_k using g".

Suppose $\alpha(c_k) = \omega$ for some c_k in V. Write $(G, \alpha)E_5(H, \beta)$ if $H = \{\eta(g, V - \{c_k\}): g \in G\}$ and $\beta = \alpha$. (That is, $g - g(c_k)c_k$ is in H if g is in G.) Then (H, β) is called the "null variable deletion of (G, α) at c_k", and (G, α) is called a "null variable augmentation of (H, β) at c_k".

Suppose V_0 is a finite nonempty subset of $\{b_k: k \geq 1\}$ such that $\eta(g, V_0) = 0$ for all g in G and $\alpha(v) = \omega$ for all v in V_0. Let G_0 be a finite (possibly empty) subset of $F_R(V)$ such that $\eta(h, V_0) = h$ for all $h \in G_0$. Write $(G, \alpha)E_6(H, \beta)$ if $H = G \cup G_0$ and $\beta(v) = \alpha(v)$ for all v not in V_0. Then (H, β) is called an "inessential variables augmentation of (G, α) by G_0 using V_0", and (G, α) is called the "inessential variables deletion of G_0 from (H, β) using V_0".

Let ξ be a "renumbering function", that is, a permutation of V such that $\xi(a_k) = a_k$ for all $k \geq 1$. Write $(G, \alpha)E_7(H, \beta)$ if $H = \{g\xi: g \in G\}$ and $\beta = \alpha\xi$. Then (H, β) is called the "renumbering of (G, α) by ξ". Observe that then (G, α) is the renumbering of (H, β) by ξ^{-1}. Let $G\xi$ denote $\{g\xi: g \in G\}$ if $G \subset F_R(V)$ and ξ is a renumbering function.

This completes the definition of $E(K; R)$. The underlying space of $M(K; R)$ is the quotient set $D(K; R)/E(K; R)$. Let $[\mathbf{s}]$ or $[G, \alpha]$ denote the equivalence class modulo $E(K; R)$ of $\mathbf{s} = (G, \alpha)$ in $D(K; R)$. For x in K, let ψ_x: $V \to K$ be the lattice constraint function given by $\psi_x(a_1) = x$ and $\psi_x(v) = \omega$ for v in $V - \{a_1\}$. Define the function ψ: $K \to M(K; R)$ by $\psi(x) = [\varnothing, \psi_x]$ for all x in K.

Interpretation. Suppose $\iota: K \to \Gamma(M; R)$ is a lattice embedding and $\iota(\omega) = 0$. One can easily verify that $(G, \alpha) \sim (H, \beta)$ in $D(K; R)$ implies $\nu(G, \alpha, \iota) = \nu(H, \beta, \iota)$ in $\Gamma_f(M^{\mathbf{N}}; R)$. That is, equivalent constraint systems have the same solution set. So, it is reasonable to treat the quotient $D(K; R)/E(K; R)$ as an abstraction of $\Gamma_f(M^{\mathbf{N}}; R)$. Furthermore, the earlier assertion that ψ abstracts ψ_M: $\Gamma(M; R) \to \Gamma_f(M^{\mathbf{N}}; R)$ is motivated by the observation that $\psi_M(\iota(x)) = \nu(\varnothing, \psi_x, \iota)$ for all x in K.

We can now begin the verification of 2.6 with a critical point. The R-constructible Horn formulas are designed to establish the next technical result. From this result it follows that ψ: $K \to M(K; R)$ is one-one if every R-constructible Horn formula is satisfied in K.

DEFINITION. For $\mathbf{s}$ and $\mathbf{t}$ in $D(K; R)$ such that $\mathbf{s}E_i\mathbf{t}$ for some i, $i \leq 4$, say that $\mathbf{t}$ is a "direct reduction" of $\mathbf{s}$. (That is, $\mathbf{t}$ is a union augmentation, defined variable augmentation, linear combination augmentation or constraint decrease of $\mathbf{s}$.) If $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_m$ for $m \geq 1$ is a sequence in $D(K; R)$ such that $\mathbf{s}_{i+1}$ is a direct reduction of $\mathbf{s}_i$ for $1 \leq i \leq m$, say that $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_m$ is a "reduction sequence" and that $\mathbf{s}_m$ is a "reduction" of $\mathbf{s}_1$. If $\mathbf{s}$ is in $D(K; R)$ and x is in K, say that s is "reducible below" x if there exists a reduction $\mathbf{t} = (H, \beta)$ of s such that $\beta(a_1) \subset x$.

3.1. *Suppose every R-constructible formula is satisfied in K. If $y, z \in K$ and $(\emptyset, \psi_y)$ in $D(K; R)$ is reducible below z, then $y \subset z$.*

PROOF. Assume the hypotheses, and let $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_m$ be a reduction sequence in $D(K; R)$ such that $\mathbf{t}_1 = (\emptyset, \psi_y)$ and $\beta_m(a_1) \subset z$, where $\mathbf{t}_i = (H_i, \beta_i)$ for $i \leqslant m$. Given h in $F_R(V)$ and B_0 with $\{b_0\} \subset B_0 \subset B = \{b_j: j \geqslant 0\}$, let $\eta_0(h, B_0)$ denote that g in $F_R(B)$ such that $g(b_0) = h(a_1)$, $g(b_j) = h(b_j)$ if $j \geqslant 1$ and $b_j \in B_0$, and $g(b_j) = 0$ otherwise. We intend to define an R-frame $\mathbf{u}_m = \langle \Psi_m, G_m, \alpha_m \rangle$ and a function $f_m: X \to K$ satisfying $\Delta(m)$ below, using induction on m.

$\Delta(m)$: The R-frame $\mathbf{u}_m = \langle \Psi_m, G_m, \alpha_m \rangle$ is the last term of some proper sequence of R-frames. If $g \in F_R(B)$ and $g \neq 0$, then $g \in G_m$ iff there exists $h \in H_m$ such that $g = \eta_0(h, \mathrm{Dom}(\alpha_m))$. For $j \geqslant 1$, if b_j is unused in G_m, then b_j is not in $\mathrm{Dom}(\alpha_m)$, and x_j does not appear in the formula Ψ_m or in any lattice polynomial $\alpha_m(b_k)$ for b_k in $\mathrm{Dom}(\alpha_m)$. Also, $f_m(x_\omega) = \omega$, $f_m(x_0) = y$, f_m satisfies the formula Ψ_m, $\bar{f}_m \alpha_m(b_0) \subset \beta_m(a_1)$ and $\bar{f}_m \alpha_m(b_k) \subset \beta_m(b_k)$ if $k \geqslant 1$ and b_k is in $\mathrm{Dom}(\alpha_m)$.

For $m = 1$, let $\mathbf{u}_1$ be the initial R-frame and define $f_1: X \to K$ by $f_1(x_0) = y$, $f_1(x_\omega) = \omega$ and $f_1(x_j) = \omega$ for $j \geqslant 1$. Then $\Delta(1)$ is satisfied.

For the induction step, assume that $m > 1$ and that $\mathbf{u}_{m-1} = \langle \Psi_{m-1}, G_{m-1}, \alpha_{m-1} \rangle$ and $f_{m-1}: X \to K$ have been defined satisfying $\Delta(m-1)$. Define $\mathbf{u}_m$ and f_m by cases, using the assumption that $\mathbf{t}_m$ is a direct reduction of $\mathbf{t}_{m-1}$.

Define $f_m = f_{m-1}$ in all cases except one. The exception occurs when $\mathbf{t}_m$ is a union augmentation of $\mathbf{t}_{m-1}$ using $c_n - b_p - b_q$ in $F_R(V)$ and z_1 and z_2 in K, and $c_n = a_1$ or $c_n = b_n$ for $n \geqslant 1$ and b_n in $\mathrm{Dom}(\alpha_{m-1})$. In that case, define f_m by $f_m(x_p) = z_1$, $f_m(x_q) = z_2$, and $f_m(x_j) = f_{m-1}(x_j)$ for $j \neq p, q$ or $x_j = x_\omega$.

In the exceptional case above, let $\mathbf{u}_m$ be the R-frame union augmentation of $\mathbf{u}_{m-1}$ using $b_0 - b_p - b_q$ if $c_n = a_1$, and let $\mathbf{u}_m$ be the union augmentation of $\mathbf{u}_{m-1}$ using $b_n - b_p - b_q$ if $c_n = b_n$ for $n \geqslant 1$ and b_n in $\mathrm{Dom}(\alpha_{m-1})$. If $\mathbf{t}_m$ is the defined variable augmentation of $\mathbf{t}_{m-1}$ using h at b_n, let $\mathbf{u}_m$ be the R-frame defined variable augmentation of $\mathbf{u}_{m-1}$ using $b_n + \eta_0(h, \mathrm{Dom}(\alpha_{m-1}))$ at b_n. If $\mathbf{t}_m$ is the linear combination augmentation of $\mathbf{t}_{m-1}$ using h and $g = \eta_0(h, \mathrm{Dom}(\alpha_{m-1})) \neq 0$, let $\mathbf{u}_m$ be the R-frame linear combination of $\mathbf{u}_{m-1}$ using g. If $\mathbf{t}_m$ is the constraint decrease of $\mathbf{t}_{m-1}$ using h at c_n and $c_n = a_1$ or $c_n = b_n$ for $n \geqslant 1$ and $b_n \in \mathrm{Dom}(\alpha_{m-1})$, let $\mathbf{u}_m$ be the R-frame constraint decrease of $\mathbf{u}_{m-1}$ using $\eta_0(h, \mathrm{Dom}(\alpha_{m-1}))$ at b_0 if $c_n = a_1$, and let $\mathbf{u}_m$ be the constraint decrease of

$\mathbf{u}_{m-1}$ at b_n using $\eta_0(h, \text{Dom}(\alpha_{m-1}))$ if $n \geq 1$ and $b_n \in \text{Dom}(\alpha_{m-1})$. In all other cases, let $\mathbf{u}_m = \mathbf{u}_{m-1}$.

We omit the verifications that the above definitions are proper and that $\Delta(m)$ is satisfied in each case, completing the induction. The Horn formula $\Psi_m \Rightarrow (x_0 = \alpha_m(b_0))$ is R-constructible by $\Delta(m)$, and so is satisfied in K by hypothesis. But f_m satisfies Ψ_m by $\Delta(m)$, so $f_m(x_0) = \bar{f}_m \alpha_m(b_0)$. But then

$$y = f_m(x_0) = \bar{f}_m \alpha_m(b_0) \subset \beta_m(a_1) \subset z,$$

using $\Delta(m)$ and our assumptions. This proves 3.1.

3.2. *If every R-constructible Horn formula is satisfied in K, then ψ: $K \to M(K; R)$ is one-one.*

PROOF. Assume the following lemma: If $\mathbf{s}$ is directly equivalent to $\mathbf{t}$ in $D(K; R)$ and $\mathbf{s}$ is reducible below some z in K, then $\mathbf{t}$ is also reducible below z. Suppose $\psi(y) = \psi(z)$ for y and z in K, so $(\varnothing, \psi_y) \sim (\varnothing, \psi_z)$ in $D(K; R)$. Now $(\varnothing, \psi_z)$ is trivially reducible below z, and so $(\varnothing, \psi_y)$ is reducible below z. (Use the lemma on a sequence of directly equivalent elements of $D(K; R)$ beginning with $(\varnothing, \psi_z)$ and ending with $(\varnothing, \psi_y)$.) But then $y \subset z$ by 3.1 and the hypotheses, and a similar argument proves $z \subset y$. Therefore, the lemma suffices to prove 3.2.

Let $\mathbf{t} = (H, \beta)$ in $D(K; R)$ and $z \in K$, and suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_m$ is a reduction sequence in $D(K; R)$, $\mathbf{s}_i = (G_i, \alpha_i)$ for $i \leq m$, such that $\alpha_m(a_1) \subset z$. To prove the lemma, we must show that $\mathbf{t}$ is reducible below z if $\mathbf{s}_1$ and $\mathbf{t}$ are related by any of the thirteen types of direct equivalence.

If $\mathbf{t}$ is a union deletion, defined variable deletion, linear combination deletion or constraint increase of $\mathbf{s}_1$, then $\mathbf{t}$ is reducible below z via the reduction sequence $\mathbf{t}, \mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_m$.

If $\mathbf{t}$ is a renumbering of $\mathbf{s}_1$ via the renumbering function ξ, then $\mathbf{t}$ is reducible below z via the reduction sequence $(G_i\xi, \alpha_i\xi)$, $i \leq m$.

For the remaining cases, a sequence $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_m$ in $D(K; R)$ and a sequence $V_1, V_2, \cdots, V_m$ of subsets of V are constructed by induction on m so that the property $\Delta_0(m)$ below is satisfied. Let $\mathbf{t}_i = (H_i, \beta_i)$ for $i \leq m$. To avoid possible conflicts of variables in the inductive constructions, we choose a sufficiently large positive integer d so that $n > d$ implies $\alpha_i(a_n) = \alpha_i(b_n) = \omega$ and $g(a_n) = g(b_n) = 0$ if $g \in G_i$, for all $i \leq m$. Recall that b_k is "unused" in G if $g \in G$ implies $g(b_k) = 0$, and b_k is "used" in G otherwise. Also, for g, h in $F_R(V)$ and $V_0 \subset V$, we write $h = \eta(g, V_0)$ if $h(v) = g(v)$ for $v \in V_0$ and $h(v) = 0$ for $v \in V - V_0$.

$\Delta_0(m)$: $\mathbf{t}_m$ is a reduction of a renumbering of $\mathbf{t}$. For some c_k in $V - (V_m \cup \{b_1, b_2, \cdots, b_d\})$ such that $\alpha_m(c_k) = \beta_m(c_k) = \omega$, each g in G_m satisfies either $\eta(g, V_m) = 0$ or $\eta(g, V_m) + rc_k$ is in H_m for some r in R. We have $a_1 \in V_m$, and $b_j \in V_m$ implies b_j is used in G_m. For $j \leqslant d$ and $c_j \in V_m$, $\beta_m(c_j) \subset \alpha_m(c_j)$. If $j \leqslant d$ and b_j is unused in G_m, then b_j is unused in H_m.

Without loss of generality, we may assume that $\mathbf{s}_1 \neq \mathbf{t}$ and that $\mathbf{t}$ is not a null variable augmentation or deletion of $\mathbf{s}_1$ at a_1. Let V' denote the set $\{a_j: j \geqslant 1\} \cup \{b_j: b_j$ is used in $G_1\}$. If $\mathbf{t}$ is a null variable deletion of $\mathbf{s}_1$ at c_n, let $\mathbf{t}_1 = \mathbf{t}$ and $V_1 = V' - \{c_n\}$. If $\mathbf{t}$ is the inessential variables deletion of G_0 from $\mathbf{s}_1$ using V_0, then let $\mathbf{t}_1 = \mathbf{t}$ and $V_1 = V' - V_0$. In the six remaining cases, $V_1 = V'$ and $\mathbf{t}_1$ is a renumbering of $\mathbf{t}$ by some ξ such that $\xi(b_k) = b_k$ if $k \leqslant d$ and b_k is used in G_1, but $\xi(b_k) = b_{u+k}$ for $k \leqslant d$ and b_k unused in G_1, where $u \geqslant d$ is sufficiently large so that $n > u$ implies $h(b_n) = 0$ for all h in H. We will omit the calculations showing that $\Delta_0(1)$ is satisfied for the above choices of $\mathbf{t}_1$ and V_1.

Assuming that $m > 1$ and that $\Delta_0(m-1)$ is satisfied for $\mathbf{t}_{m-1}$ and V_{m-1} by the induction hypothesis, we use the condition that $\mathbf{s}_m$ is a direct reduction of $\mathbf{s}_{m-1}$. If $\mathbf{s}_m$ is a union augmentation of $\mathbf{s}_{m-1}$ using $c_n - b_p - b_q$ and z_1 and z_2 in K, and $c_n \in V_{m-1}$, then let $V_m = V_{m-1} \cup \{b_p, b_q\}$ and let $\mathbf{t}_m$ be the union augmentation of $\mathbf{t}_{m-1}$ using $c_n - b_p - b_q$ and z_1 and z_2. If $\mathbf{s}_m$ is the defined variable augmentation of $\mathbf{s}_{m-1}$ using g at b_n, then let $V_m = V_{m-1} \cup \{b_n\}$ and let $\mathbf{t}_m$ be the defined variable augmentation of $\mathbf{t}_{m-1}$ using $\eta(g, V_m)$ at b_n. If $\mathbf{s}_m$ is the linear combination augmentation of $\mathbf{s}_{m-1}$ by g, and $\eta(g, V_{m-1}) \neq 0$, then let $V_m = V_{m-1}$ and let $\mathbf{t}_m$ be the linear combination augmentation of $\mathbf{t}_{m-1}$ by a suitable h of form $\eta(g, V_{m-1}) + rc_k$. If $\mathbf{s}_m$ is the constraint decrease of $\mathbf{s}_{m-1}$ using g at c_n, and $c_n \in V_{m-1}$, then let $V_m = V_{m-1}$ and let $\mathbf{t}_m$ be the constraint decrease of $\mathbf{t}_{m-1}$ by h at c_n for a suitable h of form $\eta(g, V_{m-1}) + rc_k$. In all other cases, let $\mathbf{t}_m = \mathbf{t}_{m-1}$ and $V_m = V_{m-1}$. We will omit the verifications that the above definitions of $\mathbf{t}_m$ and V_m are proper and that $\Delta_0(m)$ is satisfied in each case, completing the induction. The argument outlined above proves the lemma, since $\Delta_0(m)$ and $\alpha_m(a_1) \subset z$ imply that a renumbering of $\mathbf{t}$ is reducible below z in the final eight cases. This completes the proof of 3.2.

We make $M(K; R)$ into a lattice by operations induced in the quotient $D(K; R)/E(K; R)$. That is, binary meet and join operations are defined on $D(K; R)$ so that $E(K; R)$ is a congruence for these operations.

Interpretation. Suppose $\iota: K \to \Gamma(M; R)$ is an embedding such that $\iota(\omega) = 0$. Define $\nu_\iota: D(K; R) \to \Gamma_f(M^{\mathbf{N}}; R)$ by $\nu_\iota(\mathbf{s}) = \nu(G, \alpha, \iota)$ if $\mathbf{s} = (G, \alpha)$. Then $\mathbf{s} \wedge \mathbf{t}$ and $\mathbf{s} \vee \mathbf{t}$ are defined in $D(K; R)$ so that $\nu_\iota(\mathbf{s} \wedge \mathbf{t}) = \nu_\iota(\mathbf{s}) \cap \nu_\iota(\mathbf{t})$ and $\nu_\iota(\mathbf{s} \vee \mathbf{t}) = \nu_\iota(\mathbf{s}) + \nu_\iota(\mathbf{t})$ in $\Gamma_f(M^{\mathbf{N}}; R)$, independently of the choice of ι and M. That is, our purpose is to define formal meets and joins of constraint systems corresponding to the usual meets and joins of their solution sets.

DEFINITION. For s = (G, α) in $D(K; R)$, let the "length" of s, denoted $|s|$, equal the smallest integer n, $n \geq 1$, such that $k > n$ implies $\alpha(a_k) = \alpha(b_k) = \omega$ and $g(a_k) = g(b_k) = 0$ for all g in G. Clearly, $|s|$ exists for every s in $D(K; R)$.

For c_j, c_k in V, let $\mathrm{tr}(c_j, c_k)$: $V \to V$ denote the bijection transposing c_j and c_k and leaving all other v in V fixed. If $c_j = c_k$, $\mathrm{tr}(c_j, c_k) = 1_V$. Given $m \geq 1$ and $i, j \geq 0$, we will denote by $\mathrm{tr}_m(i, j)$ the product $\Pi_{k=1}^{m} \mathrm{tr}(b_{im+k}, b_{jm+k})$, where the product operation is composition of functions. These functions $\mathrm{tr}_m(i, j)$ are frequently used as renumbering functions. We also define bijections $\theta_{m,n}$: $V \to V$ given by the product $\Pi_{k=1}^{m} \mathrm{tr}(b_k, b_{2nm+k})\mathrm{tr}(a_k, b_{(2n+1)m+k})$, for any $m \geq 1$ and $n \geq 0$. For (G, α) in $D(K; R)$, we let $G\theta_{m,n}$ denote the finite subset $\{g\theta_{m,n}: g \in G\}$ of $F_R(V)$, and observe that $\alpha\theta_{m,n}$ is a lattice constraint function. (We will also let $-G$ denote the set $\{-g: g \in G\}$ for $G \subset F_R(V)$.)

For $m \geq 1$ and $i_1, i_2, \cdots, i_n \geq 0$, let $Q_{i_1, i_2, \cdots, i_n}(m)$ denote the following m-subset of $F_R(V)$:

$$\left\{a_k - \sum_{j=1}^{n} b_{i_j m+k}: k = 1, 2, \cdots, m\right\}.$$

For example, $Q_{1,3}(m) = \{a_k - b_{m+k} - b_{3m+k}: k \leq m\}$.

If α, β: $V \to K$ are lattice constraint functions, let $\alpha \vee \beta$ and $\alpha \wedge \beta$ be the lattice constraint functions given by $(\alpha \vee \beta)(v) = \alpha(v) \vee \beta(v)$ and $(\alpha \wedge \beta)(v) = \alpha(v) \wedge \beta(v)$ for all v in V. Let α^a and α^b be functions given as follows: $\alpha^a(a_k) = \alpha(a_k)$, $\alpha^a(b_k) = \omega$, $\alpha^b(a_k) = \omega$ and $\alpha^b(b_k) = \alpha(b_k)$, for all $k \geq 1$.

Let s = (G, α) and t = (H, β) in $D(K; R)$, and suppose that $m \geq |s|, |t|$. Define $y_m(s, t) = (J_1, \gamma_1)$ and $z_m(s, t) = (J_2, \gamma_2)$ in $D(K; R)$ as follows:

$$J_1 = Q_{1,3}(m) \cup G\theta_{m,0} \cup H\theta_{m,1}, \qquad \gamma_1 = (\alpha \vee \beta)^a \vee a\theta_{m,0} \vee \beta\theta_{m,1},$$

$$J_2 = Q_1(m) \cup Q_3(m) \cup G\theta_{m,0} \cup H\theta_{m,1}, \gamma_2 = (\alpha \wedge \beta)^a \vee \alpha\theta_{m,0} \vee \beta\theta_{m,1}.$$

Define s $\vee$ t $= y_n(s, t)$ and s $\wedge$ t $= z_n(s, t)$ for $n = \max\{|s|, |t|\}$. Note that $|y_m(s, t)| = |z_m(s, t)| = 4m$ for $m \geq |s|, |t|$.

Interpretation. Suppose ι: $K \to M(K; R)$ is an embedding and $\iota(\omega) = 0$. Let s = (G, α) and t = (H, β) in $D(K; R)$, and let $n = \max\{|s|, |t|\}$. The extended solutions h: $V \to M$ in $\nu_0(G, \alpha, \iota)$ or in $\nu_0(H, \beta, \iota)$ can be represented in tabular form by pairs of sequences (h_a, h_b), where

$$h_a = (h(a_1), h(a_2), \cdots, h(a_n), 0, 0, \cdots), \quad \text{and}$$

$$h_b = (h(b_1), h(b_2), \cdots, h(b_n), 0, 0, \cdots).$$

(Note that $h(a_k) = h(b_k) = 0$ for $k > n$ because $\iota(\omega) = 0$ and $n \geq |s|, |t|$.) The solution corresponding to this extended solution h can be represented by the first sequence h_a. Now $\mathbf{s} \vee \mathbf{t} = (J_1, \gamma_1)$ is constructed so that $f\colon V \to M$ is in $\nu_0(J_1, \gamma_1, \iota)$ if and only if there exist $g\colon V \to M$ in $\nu_0(G, \alpha, \iota)$ and $h\colon V \to M$ in $\nu_0(H, \beta, \iota)$ such that

$$f_a = (g(a_1) + h(a_1), g(a_2) + h(a_2), \cdots, g(a_n) + h(a_n), 0, 0, \cdots),$$
$$f_b = (g(b_1), g(b_2), \cdots, g(b_n), g(a_1), g(a_2), \cdots, g(a_n),$$
$$h(b_1), h(b_2), \cdots, h(b_n), h(a_1), h(a_2), \cdots, h(a_n), 0, 0, \cdots).$$

So, the solutions f_a correspond precisely to sums of solutions $g_a + h_a$, as was required. Similarly, for $\mathbf{s} \wedge \mathbf{t} = (J_2, \gamma_2)$, $f\colon V \to M$ is in $\nu_0(J_2, \gamma_2, \iota)$ if and only if there exist g in $\nu_0(G, \alpha, \iota)$ and h in $\nu_0(H, \beta, \iota)$ such that $f_a = g_a = h_a$ and f_b is again given by the formula above. Therefore, the solutions f_a corresponding to $\mathbf{s} \wedge \mathbf{t}$ are just the solutions common to $\mathbf{s}$ and $\mathbf{t}$, as was required.

3.3. *For* $\mathbf{s}, \mathbf{t}$ *in* $D(K; R)$, $\mathbf{s} \vee \mathbf{t} \sim \mathbf{t} \vee \mathbf{s}$ *and* $\mathbf{s} \wedge \mathbf{t} \sim \mathbf{t} \wedge \mathbf{s}$.

PROOF. Clearly $\mathbf{t} \vee \mathbf{s}$ is the renumbering of $\mathbf{s} \vee \mathbf{t}$ using $\mathrm{tr}_{2n}(0, 1)$ for $n = \max\{|\mathbf{s}|, |\mathbf{t}|\}$, and similarly $\mathbf{t} \wedge \mathbf{s}$ is the renumbering of $\mathbf{s} \wedge \mathbf{t}$ using $\mathrm{tr}_{2n}(0, 1)$.

3.4. *Let* $(G \cup J, \alpha)$ *and* $(H \cup J, \alpha)$ *be in* $D(K; R)$, *and suppose that for every* g *in* G *there exists* h *in* H *such that* $g(c_k) = h(c_k)$ *for all* c_k *in* V *such that* $\alpha(c_k) \neq \omega$, *and that for every* h *in* H *there exists* g *in* G *such that* $g(c_k) = h(c_k)$ *for all* c_k *in* V *such that* $\alpha(c_k) \neq \omega$. *Then* $(G \cup J, \alpha) \sim (H \cup J, \alpha)$ *in* $D(K; R)$. *In particular, if* g *in* G *and* $g(c_k) \neq 0$ *implies* $\alpha(c_k) = \omega$ *for all* c_k *in* V, *then* $(G \cup J, \alpha) \sim (J, \alpha)$. *Furthermore, if* $\alpha^a = \psi_\omega$, *then* $(J, \alpha) \sim (\varnothing, \psi_\omega)$.

PROOF. Assume the hypotheses for $(G \cup J, \alpha)$ and $(H \cup J, \alpha)$, and choose $n \geqslant |(G \cup J, \alpha)|, |(H \cup J, \alpha)|$. Perform null variable deletions at c_k for all values of $k \leqslant n$ such that $\alpha(c_k) = \omega$. So, we obtain $(G \cup J, \alpha) \sim (G_0 \cup J_0, \alpha)$ and $(H \cup J, \alpha) \sim (H_0 \cup J_0, \alpha)$, where G_0 is the set of equations obtained from equations g of G by dropping all nonzero terms $g(c_k)c_k$ for which $\alpha(c_k) = \omega$, and H_0 and J_0 are obtained from H and J, respectively, in the same way. By the hypotheses on G and H, however, we have $G_0 = H_0$, and so $(G \cup J, \alpha) \sim (H \cup J, \alpha)$.

Suppose that $g(c_k) \neq 0$ implies $\alpha(c_k) = \omega$ for all g in G and c_k in V. Then $(G \cup J, \alpha) \sim (\{0\} \cup J, \alpha)$ as above, for $G \neq \varnothing$. If J is nonempty, then $(\{0\} \cup J, \alpha) \sim (J, \alpha)$ by a linear combination deletion. It is also easily checked that $(\{0\}, \alpha) \sim (\varnothing, \alpha)$, so $(G \cup J, \alpha) \sim (J, \alpha)$.

Finally, suppose $\alpha^a = \psi_\omega$. Noting that $(J, \alpha) \sim (J_0, \alpha)$ by null variable deletions as above, we observe that h in J_0 implies $h(a_k) = 0$ for $k \geqslant 1$ because $\alpha^a = \psi_\omega$. Also, $h(b_k) = 0$ if $k > m = |(J_0, \alpha)|$. But then $(J_0, \alpha) \sim (\varnothing, \psi_\omega)$ by an inessential variables deletion of J_0 using $\{b_1, b_2, \cdots, b_m\}$. This completes the proof of 3.4.

3.5. *If* s, t *are in* $D(K; R)$ *and* $m \geqslant |\mathbf{s}|, |\mathbf{t}|$, *then* $\mathbf{s} \vee \mathbf{t} \sim y_m(\mathbf{s}, \mathbf{t})$ *and* $\mathbf{s} \wedge \mathbf{t} \sim z_m(\mathbf{s}, \mathbf{t})$.

PROOF. Assume the hypotheses, and let $n = \max\{|\mathbf{s}|, |\mathbf{t}|\}$. If $y_m(\mathbf{s}, \mathbf{t}) = (J, \gamma)$, then $\gamma(a_k) = \gamma(b_{m+k}) = \gamma(b_{3m+k}) = \omega$ for $n < k \leqslant m$. If $J_1 = \{a_k - b_{m+k} - b_{3m+k}: n < k \leqslant m\}$, then $(J, \gamma) \sim (J - J_1, \gamma)$ by 3.4. But $\mathbf{s} \vee \mathbf{t} = y_n(\mathbf{s}, \mathbf{t})$ is clearly a renumbering of $(J - J_1, \gamma)$, so $\mathbf{s} \vee \mathbf{t} \sim y_m(\mathbf{s}, \mathbf{t})$. The proof that $\mathbf{s} \wedge \mathbf{t} \sim z_m(\mathbf{s}, \mathbf{t})$ is similar.

3.6. $E(K; R)$ *is a congruence for meet and join in* $D(K; R)$.

PROOF. Suppose s, t, u are in $D(K; R)$, and $n = \max\{|\mathbf{s}|, |\mathbf{t}|, |\mathbf{u}|\}$. It suffices to prove the following lemma: If $\mathbf{s}E_j\mathbf{t}$ for some j, $j \leqslant 7$, then $y_m(\mathbf{s}, \mathbf{u}) \sim y_m(\mathbf{t}, \mathbf{u})$ and $z_m(\mathbf{s}, \mathbf{u}) \sim z_m(\mathbf{t}, \mathbf{u})$ for some m, $m \geqslant n$. From this lemma and 3.5, an induction argument proves that $\mathbf{s} \sim \mathbf{t}$ implies $\mathbf{s} \wedge \mathbf{u} \sim \mathbf{t} \wedge \mathbf{u}$ and $\mathbf{s} \vee \mathbf{u} \sim \mathbf{t} \vee \mathbf{u}$. Using this result and 3.3, it follows that $E(K; R)$ is a congruence.

If t is a constraint decrease of s at a_k using g, then $y_n(\mathbf{t}, \mathbf{u})$ can be otained from $y_n(\mathbf{s}, \mathbf{u})$ by two constraint decreases (at b_{n+k} using $g\theta_{n,0}$, then at a_k using $a_k - b_{n+k} - b_{3n+k}$).

If t is a null variable deletion of s at a_k and $m \geqslant k, n$, then $y_m(\mathbf{t}, \mathbf{u})$ can be obtained from $y_m(\mathbf{s}, \mathbf{u})$ by a null variable deletion at b_{m+k} followed by a null variable augmentation at b_{m+k} restoring the linking equation $a_k - b_{m+k} - b_{3m+k}$.

If t is a renumbering of s by a renumbering function ξ, then we can suppose without loss of generality that $\xi(b_k) = b_k$ for all $k > n$. But then $y_n(\mathbf{t}, \mathbf{u})$ is a renumbering of $y_n(\mathbf{s}, \mathbf{u})$ using ξ also.

In the remaining cases, it is relatively easy to show that $\mathbf{s}E_j\mathbf{t}$ implies $y_m(\mathbf{s}, \mathbf{u}) E_j y_m(\mathbf{t}, \mathbf{u})$ for some sufficiently large m. We will omit these cases, plus all the similar arguments showing that $\mathbf{s}E_j\mathbf{t}$ implies $z_m(\mathbf{s}, \mathbf{u}) \sim z_m(\mathbf{t}, \mathbf{u})$. This outlines the proof of the lemma, completing the proof of 3.6.

DEFINITION. If $G_i \subset F_R(V)$ for $i = 1, 2, \cdots, n$, let $[[G_1, G_2, \cdots, G_n]]$ denote the R-submodule of $F_R(V)$ generated by $G_1 \cup G_2 \cup \cdots \cup G_n$.

3.7. *If* $(G \cup H, \alpha)$ *and* $(G \cup J, \alpha)$ *are in* $D(K; R)$ *such that* $[[G, H]] = [[G, J]]$, *then* $(G \cup H, \alpha) \sim (G \cup J, \alpha)$. *In particular,* $(G \cup H, \alpha) \sim (G \cup (-H), \alpha)$.

PROOF. Assuming the hypotheses, both $(G \cup H, \alpha)$ and $(G \cup J, \alpha)$ are equivalent to $(G \cup H \cup J, \alpha)$ by some number of linear combination augmentations. For the second part, observe that $[[G, H]] = [[G, -H]]$.

DEFINITION. Suppose $J_0, J_1, \cdots, J_p$ and $G_0, G_1, \cdots, G_q$ are finite subsets of $F_R(V)$, and $\alpha, \alpha_0, \alpha_1, \cdots, \alpha_q$ are lattice constraint functions. For

$m \geq 1$, $F_m(J_0, J_1, \cdots, J_p | G_0, G_1, \cdots, G_q)$ denotes

$$J_0 \cup J_1 \cup \cdots \cup J_p \cup G_0\theta_{m,0} \cup G_1\theta_{m,1} \cup \cdots \cup G_q\theta_{m,q},$$

and $f_m(\alpha | \alpha_0, \alpha_1, \cdots, \alpha_q)$ denotes $\alpha \vee \alpha_0\theta_{m,0} \vee \alpha_1\theta_{m,1} \vee \cdots \vee \alpha_q\theta_{m,q}$. In general, we will use these notations only when $m \geq |(G_i, \alpha_i)|$ for $0 \leq i \leq q$, and $\alpha = \alpha^a$. In this case, we observe that h in $G_i\theta_{m,i}$ for some i, $0 \leq i \leq q$, implies that $h(a_k) = 0$ for all $k \geq 1$, and $h(b_k) = 0$ unless $2im < k \leq (2i+2)m$. Furthermore, $\gamma = f_m(\alpha | \alpha_0, \alpha_1, \cdots, \alpha_q)$ is then given by $\gamma(a_k) = \alpha(a_k)$ for all $k \geq 1$, $\gamma(b_{2im+k}) = \alpha_i(b_k)$ and $\gamma(b_{(2i+1)m+k}) = \alpha_i(a_k)$ for $k \leq m$ and $0 \leq i \leq q$, and $\gamma(b_k) = \omega$ for $k > (2q+2)m$.

Suppose $i_1, i_2, \cdots, i_p$ and $j_1, j_2, \cdots, j_q$ are nonnegative integers, and $m \geq 1$. Define $P^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_q}(m)$ as the following subset of $F_R(V)$:

$$\left\{\left(\sum_{k=1}^{p} b_{i_k m+d}\right) - \left(\sum_{k=1}^{q} b_{j_k m+d}\right) : d = 1, 2, \cdots, m\right\}.$$

For example, $P^3_{7,9}(m) = \{b_{3m+d} - b_{7m+d} - b_{9m+d} : d \leq m\}$. In the following, we will often let

$$Q_{j_1, j_2, \cdots, j_q} \quad \text{and} \quad P^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_q}$$

abbreviate

$$Q_{j_1, j_2, \cdots, j_q}(m) \quad \text{and} \quad P^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_q}(m),$$

respectively, when m can be understood from the context. In particular, m is understood for an argument of the F_m notation.

3.8. *Let $H, G_0, G_1, \cdots, G_n$ be finite subsets of $F_R(V)$, let $m \geq |(G_i, \psi_\omega)|$ for $0 \leq i \leq n$, and let $r_1, r_2, r_3 \in R$ such that $r_1 \in \{1, -1\}$. Suppose i, j and k are distinct integers, $0 \leq i, j, k \leq n$, such that the following equations are in H:*

$$r_1 b_{2im+d} + r_2 b_{2jm+d} + r_3 b_{2km+d} \text{ whenever } d \leq m \text{ and } g(b_d) \neq 0 \text{ for some } g \text{ in } G_j, \text{ and}$$

$$r_1 b_{(2i+1)m+d} + r_2 b_{(2j+1)m+d} + r_3 b_{(2k+1)m+d} \text{ whenever } d \leq m \text{ and } g(a_d) \neq 0 \text{ for some } g \text{ in } G_j.$$

Furthermore, suppose either that $r_3 = 0$ or that $G_j \subset G_k$. Then $G_j\theta_{m,i} \subset [[F_m(H|G_0, G_1, \cdots, G_n)]]$. If in addition $r_2 \in \{1, -1\}$, and $H_j = \varnothing$, $H_i = G_i \cup G_j$ and $H_p = G_p$ for $p \neq i, j$, then

$$[[F_m(H|G_0, G_1, \cdots, G_n)]] = [[F_m(H|H_0, H_1, \cdots, H_n)]].$$

If $G, J, J_0, J_1, \cdots, J_n$ are finite subsets of $F_R(V)$ such that $m \geqslant |(J, \psi_\omega)|$ and $Q_1(m) \subset G$, then

$$[[F_m(G, J|J_0, J_1, \cdots, J_n)]] = [[F_m(G|J \cup J_0, J_1, \cdots, J_n)]].$$

PROOF. Assume the hypotheses of the first part, and observe that $g \in G_j$ implies

$$r_1 g\theta_{m,i} + r_2 g\theta_{m,j} + r_3 g\theta_{m,k}$$
$$- \sum_{d=1}^{m} g(b_d)(r_1 b_{2im+d} + r_2 b_{2jm+d} + r_3 b_{2km+d})$$
$$- \sum_{d=1}^{m} g(a_d)(r_1 b_{(2i+1)m+d} + r_2 b_{(2j+1)m+d} + r_3 b_{(2k+1)m+d}) = 0.$$

But $r_3 g\theta_{m,k} \in [[G_k\theta_{m,k}]]$ because either $r_3 = 0$ or $G_j \subset G_k$, by hypothesis. So, $G_j\theta_{m,i} \subset [[H, G_j\theta_{m,j}, G_k\theta_{m,k}]]$ because $r_1 \in \{1, -1\}$, and

$$[[H, G_k\theta_{m,k}, G_j\theta_{m,j}]] = [[H, G_k\theta_{m,k}, G_j\theta_{m,i}]]$$

if we also assume $r_2 \in \{1, -1\}$. This proves the first part.

For the second part, observe that $g \in J$ implies that

$$g - g\theta_{m,0} - \sum_{d=1}^{m} g(a_d)(a_d - b_{m+d}) = 0.$$

Therefore, $[[G, J]] = [[G, J\theta_{m,0}]]$ if $Q_1(m) \subset G$, and the result follows. This completes the proof of 3.8.

DEFINITION. We will use reverse functional notation for lattice polynomials $w(s_1, s_2, \cdots, s_q)$ with variables s_i, $i \leqslant q$, in $D(K; R)$. That is, a "$D(K; R)$ polynomial" is recursively defined as (1) a one-term sequence (x_0) with $x_0 \in D(K; R)$, or (2) a sequence formed from shorter $D(K; R)$ polynomials e and e_0 by juxtaposing sequences $e * e_0 * (x_n)$, where $x_n \in \{\vee, \wedge\}$. (For example, $(s_1, s_2, \vee, s_3, s_1, \vee, \wedge)$ is the $D(K; R)$ polynomial usually written as $(s_1 \vee s_2) \wedge (s_3 \vee s_1)$.) Given a $D(K; R)$ polynomial $w(s_1, s_2, \cdots, s_q) = (x_0, x_1, \cdots, x_n)$, there is a unique $D(K; R)$ polynomial $w_i(s_1, s_2, \cdots, s_q)$ of form $(x_j, x_{j+1}, \cdots, x_i)$ for each i, $0 \leqslant i \leqslant n$. If $x_i = s_p$ in $D(K; R)$, then $w_i(s_1, s_2, \cdots, s_q) = (x_i)$. If $x_i \in \{\vee, \wedge\}$, however, then there exists a unique integer k, $j < k < i$, such that

$$w_{k-1}(s_1, s_2, \cdots, s_q) = (x_j, x_{j+1}, \cdots, x_{k-1}),$$
$$w_{i-1}(s_1, s_2, \cdots, s_q) = (x_k, x_{k+1}, \cdots, x_{i-1}) \quad \text{and}$$
$$w_i(s_1, s_2, \cdots, s_q) = w_{k-1}(s_1, s_2, \cdots, s_q) x_i w_{i-1}(s_1, s_2, \cdots, s_q).$$

In this case, we say that x_i is "connected" to x_{k-1} and x_{i-1}. Of course, $w_n(s_1, s_2, \cdots, s_q) = w(s_1, s_2, \cdots, s_q)$. We observe that x_0 is in $D(K; R)$ always, and that $x_n \in \{\vee, \wedge\}$ if $n > 0$. Furthermore, there is a unique "path" from x_n to each x_i, $i < n$; that is, there is a sequence $x_{i_1}, x_{i_2}, \cdots, x_{i_p}$ such that $x_{i_1} = x_n$, $x_{i_p} = x_i$ and x_{i_j} is connected to $x_{i_{j+1}}$ for $1 \leqslant j < p$.

Suppose $w(s_1, s_2, \cdots, s_q) = (x_0, x_1, \cdots, x_n)$ is a $D(K; R)$ polynomial, for $s_i = (G_i, \alpha_i)$ in $D(K; R)$, $i \leqslant q$. We will recursively define $\mathbf{F}_m(x_0, x_1, \cdots, x_n)$ in $D(K; R)$, for any $m \geqslant |s_1|, |s_2|, \cdots, |s_q|$. If $x_i = s_p$ for some $p \leqslant q$, then define $H_i = G_p$ and $\beta_i = \alpha_p$. If $x_i \in \{\vee, \wedge\}$, then let $H_i = \varnothing$ and $\beta_i = \psi_\omega$. We now define $\mathbf{F}_m(x_0, x_1, \cdots, x_n) = (J, \gamma)$ as follows:

$$J = F_m(H|H_0, H_1, \cdots, H_n),$$

$$\gamma = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)^a|\beta_0, \beta_1, \cdots, \beta_n).$$

It remains to give the definition of H. For any finite set I containing sequences $\iota = (i_1, i_2, \cdots, i_p)$ of nonnegative integers, let $Q(I)$ denote $\bigcup_{\iota \in I} Q_\iota$. (For example, if $I = \{(1), (5, 7)\}$, then $Q(I) = Q_1 \cup Q_{5,7}$.) If I and I' are sets of sequences of integers, let $I * I'$ denote the juxtaposition set product $\{\iota * \iota' : \iota \in I, \iota' \in I'\}$. (For example, $I * I' = \{(1, 5), (1, 7, 9), (3, 5), (3, 7, 9)\}$ if $I = \{(1), (3)\}$ and $I' = \{(5), (7, 9)\}$.) We now define a sequence $I_0, I_1, \cdots, I_n$ of such sets recursively, using $(x_0, x_1, \cdots, x_n)$. If x_i is in $D(K; R)$, let $I_i = \{(2i + 1)\}$. Otherwise, $x_i \in \{\vee, \wedge\}$, so x_i is connected to x_{k-1} and x_{i-1} for a unique $k < i$. In that case, define $I_i = I_{k-1} * I_{i-1}$ if x_i equals $\vee$, and define $I_i = I_{k-1} \cup I_{i-1}$ if x_i equals $\wedge$. This uniquely defines $I_0, I_1, \cdots, I_n$, and the definition of $\mathbf{F}_m(x_0, x_1, \cdots, x_n)$ is completed by setting $H = Q(I_n)$.

EXAMPLE. Suppose $w(s_1, s_2, s_3) = (s_1, s_2, \vee, s_3, s_1, \vee, \wedge)$, where $s_i = (G_i, \alpha_i)$ for $i = 1, 2, 3$. By our definitions, $I_0 = \{(1)\}$, $I_1 = \{(3)\}$, $I_2 = I_0 * I_1 = \{(1, 3)\}$, $I_3 = \{(7)\}$, $I_4 = \{(9)\}$, $I_5 = I_3 * I_4 = \{(7, 9)\}$ and $I_6 = I_2 \cup I_5 = \{(1, 3), (7, 9)\}$. Therefore, $Q(I_6) = Q_{1,3} \cup Q_{7,9}$, so $\mathbf{F}_m(s_1, s_2, \vee, s_3, s_1, \vee, \wedge) = (J, \gamma)$ as given below:

$$J = F_m(Q_{1,3}, Q_{7,9}|G_1, G_2, \varnothing, G_3, G_1), \quad \text{and}$$

$$\gamma = f_m(((\alpha_1 \vee \alpha_2) \wedge (\alpha_3 \vee \alpha_1))^a|\alpha_1, \alpha_2, \psi_\omega, \alpha_3, \alpha_1).$$

Note that we have replaced the union symbol in the expression for $Q(I_6)$ by a comma, as our F_m notation permits. We will replace union symbols by commas similarly in subsequent evaluations of $\mathbf{F}_m$ applied to $D(K; R)$ polynomials. We will also delete final null terms, as $H_5 = H_6 = \varnothing$ and $\beta_5 = \beta_6 = \psi_\omega$ were deleted above.

3.9. *If* $w(s_1, s_2, \cdots, s_q) = (x_0, x_1, \cdots, x_n)$ *is a* $D(K; R)$ *polynomial and* $m \geqslant |s_1|, |s_2|, \cdots, |s_q|$, *then*

$$w(\mathrm{s}_1, \mathrm{s}_2, \cdots, \mathrm{s}_q) \sim \mathrm{F}_m(x_0, x_1, \cdots, x_n), \quad \textit{in } D(K; R).$$

PROOF. Assume the hypothesis, and let $\mathrm{s}_i = (G_i, \alpha_i)$ for $i \leq q$. Let $\mathrm{F}_m(x_0, x_1, \cdots, x_n) = (J_0, \gamma_0)$ for $J_0 = F_m(Q(I_n)|H_0, H_1, \cdots, H_n)$ and $\gamma_0 = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)^a|\beta_0, \beta_1, \cdots, \beta_n)$, where the sequences $(H_0, H_1, \cdots, H_n)$, $(\beta_0, \beta_1, \cdots, \beta_n)$ and $(I_0, I_1, \cdots, I_n)$ are formed as described above. Define a sequence $U_0, U_1, \cdots, U_n$ of subsets of $F_R(V)$ and a sequence $\kappa_0, \kappa_1, \cdots, \kappa_n$ of lattice constraint functions as follows: If $x_i = \mathrm{s}_p$ in $D(K;R)$, then $U_i = H_i = G_p$ and $\kappa_i = \beta_i = \alpha_p$. If $x_i \in \{\vee, \wedge\}$, then x_i is connected to x_{k-1} and x_{i-1} for a unique $k < i$. In this case, $\kappa_i = w_i(\alpha_1, \alpha_2, \cdots, \alpha_q)^a$, and $U_i = Q_{2k-1,2i-1}$ if x_i equals $\vee$, and $U_i = Q_{2k-1} \cup Q_{2i-1}$ if x_i equals $\wedge$. Define a sequence $(J_0, \gamma_0), (J_1, \gamma_1), \cdots, (J_n, \gamma_n)$ in $D(K; R)$ as follows:

$$J_i = F_m(Q(I_n)|U_0, U_1, \cdots, U_i, H_{i+1}, H_{i+2}, \cdots, H_n),$$

$$\gamma_i = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)^a|\kappa_0, \kappa_1, \cdots, \kappa_i, \beta_{i+1}, \beta_{i+2}, \cdots, \beta_n).$$

(Since $x_0 \in D(K; R)$, the two definitions of (J_0, γ_0) agree.) We intend to prove that $(J_{i-1}, \gamma_{i-1}) \sim (J_i, \gamma_i)$ in $D(K; R)$ for $0 < i \leq n$.

Consider first the following lemma: For $0 \leq j \leq n$, if $\iota = (i_1, i_2, \cdots, i_p) \in I_j$, and $M_j = [[F_m(\emptyset|U_0, U_1, \cdots, U_j)]]$, then $P_\iota^{2j+1} \subset M_j$. Using induction on j, we note that $I_j = \{(2j+1)\}$ if x_j is in $D(K; R)$, and so $P_\iota^{2j+1} = \{0\}$ for ι in I_j in this case, and in the case $j = 0$ in particular. Assuming the induction hypothesis, suppose $x_j \in \{\vee, \wedge\}$, where x_j is connected to x_{k-1} and x_{j-1}, and $\iota \in I_j$. If x_j equals $\vee$, then $\iota = \iota_1 * \iota_2$ for some ι_1 in I_{k-1} and ι_2 in I_{j-1}. Furthermore, $P_{2k-1,2j-1}^{2j+1} = U_j\theta_{m,j} \subset M_j$, and $P_{\iota_1}^{2k-1} \subset M_{k-1} \subset M_j$ and $P_{\iota_2}^{2j-1} \subset M_{j-1} \subset M_j$ by the induction hypothesis. Therefore

$$P_\iota^{2j+1} \subset [[P_{2k-1,2j-1}^{2j+1}, P_{\iota_1}^{2k-1}, P_{\iota_2}^{2j-1}]] \subset M_j \quad \text{if } x_j \text{ equals } \vee.$$

Finally, suppose x_j equals $\wedge$, so $\iota \in I_{k-1} \cup I_{j-1}$. Observing that $P_{2k-1}^{2j+1} \cup P_{2j-1}^{2j+1} = U_j\theta_{m,j} \subset M_j$, we see that

$$P_\iota^{2j+1} \subset [[P_{2k-1}^{2j+1}, P_\iota^{2k-1}]] \subset M_j \quad \text{or} \quad P_\iota^{2j+1} \subset [[P_{2j-1}^{2j+1}, P_\iota^{2j-1}]] \subset M_j,$$

using the induction hypothesis. This completes the proof of the lemma.

If x_i is in $D(K; R)$, then $(J_{i-1}, \gamma_{i-1}) = (J_i, \gamma_i)$. So, assume that $x_i \in \{\vee, \wedge\}$, and let k be the unique integer, $0 < k < i$, such that x_i is connected to x_{k-1} and x_{i-1}. If x_i equals $\vee$, then make defined variable augmentations at $b_{(2i+1)m+d}$ for $d \leq m$ using the equations $P_{2k-1,2i-1}^{2i+1}(m)$. Since $P_{2k-1,2i-1}^{2i+1} = Q_{2k-1,2i-1}\theta_{m,i} = U_i\theta_{m,i}$ and $\kappa_i = w_i(\alpha_1, \alpha_2, \cdots, \alpha_q)^a = w_{k-1}(\alpha_1, \alpha_2, \ldots, \alpha_q)^a \vee w_{i-1}(\alpha_1, \alpha_2, \cdots, \alpha_q)^a = (\kappa_{k-1})^a \vee (\kappa_{i-1})^a$, the above argument proves $(J_{i-1}, \gamma_{i-1}) \sim (J_i, \gamma_i)$. Now suppose that x_i equals $\wedge$. We make defined variable augmentations at $b_{(2i+1)m+d}$ for $d \leqslant m$ using $P^{2i+1}_{2k-1}(m)$. Since $P^{2i+1}_{2k-1} = Q_{2k-1}\theta_{m,i}$, we obtain $(J_{i-1}, \gamma_{i-1}) \sim (J_i^*, \gamma_i^*)$ as given below:

$$J_i^* = F_m(Q(I_n)|U_0, U_1, \cdots, U_{i-1}, Q_{2k-1}, H_{i+1}, H_{i+2}, \cdots, H_n),$$

$$\gamma_i^* = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)^a|\kappa_0, \kappa_1, \cdots, \kappa_{i-1}, (\kappa_{k-1})^a, \beta_{i+1}, \beta_{i+2}, \cdots, \beta_n).$$

We now prove that $P^{2i+1}_{2i-1} \subset [[J_i^*]]$. Clearly $I_0, I_1, \cdots, I_n$ are nonempty sets, so we choose sequences ι_1 in I_{k-1} and ι_2 in I_{i-1} and note that $\{\iota_1, \iota_2\} \subset I_i$. Since $i = n$ or there is a "path" from x_n to x_i, we can find (possibly empty) sequences ι_3 and ι_4 such that $\iota_5 = \iota_3 * \iota_1 * \iota_4$ and $\iota_6 = \iota_3 * \iota_2 * \iota_4$ are in I_n. Now, $P^{\iota_1}_{\iota_2} \subset [[Q_{\iota_5}, Q_{\iota_6}]] \subset [[Q(I_n)]] \subset [[J_i^*]]$. (For example, $P^3_{5,7} \subset [[Q_{1,3,9}, Q_{1,5,7,9}]]$.) But then

$$P^{2i+1}_{2i-1} \subset [[P^{2k-1}_{\iota_1}, P^{2i-1}_{\iota_2}, P^{\iota_1}_{\iota_2}, P^{2i+1}_{2k-1}]] \subset [[J_i^*]],$$

using the lemma and the equation $Q_{2k-1}\theta_{m,i} = P^{2i+1}_{2k-1}$. So $(J_i^*, \gamma_i^*) \sim (J_i, \gamma_i)$ by linear combination augmentations adding the equations $P^{2i+1}_{2i-1} = Q_{2i-1}\theta_{m,i}$, followed by constraint decreases at $b_{(2i+1)m+d}$ for $d \leqslant m$ using these equations, since $\kappa_i = w_i(\alpha_1, \alpha_2, \cdots, \alpha_q)^a = w_{k-1}(\alpha_1, \alpha_2, \cdots, \alpha_q)^a \wedge w_{i-1}(\alpha_1, \alpha_2, \cdots, \alpha_q)^a = (\kappa_{k-1})^a \wedge (\kappa_{i-1})^a$. This completes the proof that $(J_{i-1}, \gamma_{i-1}) \sim (J_i, \gamma_i)$ for $0 < i \leqslant n$.

Let $J = F_m(Q_{2n+1}|U_0, U_1, \cdots, U_n)$, and recall that

$$J_n = F_m(Q(I_n)|U_0, U_1, \cdots, U_n).$$

For ι in I_n, note that $P^{2n+1}_\iota \subset F_m(\varnothing|U_0, U_1, \cdots, U_n)$ by the lemma, and also that $[[Q_{2n+1}, P^{2n+1}_\iota]] = [[Q_\iota, P^{2n+1}_\iota]]$. But then $[[J]] = [[J_n]]$ follows, and so $(J_n, \gamma_n) \sim (J, \gamma_n)$ by 3.7.

Finally, we prove that $(J, \gamma_n) \sim w(\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_q)$ by induction on n. If $n = 0$, then $(x_0) = (\mathbf{s}_j)$ and $w(\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_q) = \mathbf{s}_j$ for some j. So, $J = F_m(Q_1|G_j)$ and $\gamma_n = (\alpha_j^a|\alpha_j)$, and $(J, \gamma_n) \sim (J', \gamma_n)$ for $J' = F_m(Q_1, G_j|\varnothing)$ by 3.8 and 3.7. Replacing Q_1 by $-Q_1$ using 3.7, and eliminating $-Q_1$ by defined variable deletions at b_{m+d} for $d \leqslant m$, we obtain $(J', \gamma_n) \sim \mathbf{s}_j$. This proves $(J, \gamma_n) \sim w(\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_q)$ if $n = 0$.

Now suppose that $n > 0$, so $x_n \in \{\vee, \wedge\}$ and x_n is connected to x_{k-1} and x_{n-1} for a unique k, $0 < k < n$. That is, $w = w_{k-1}x_n w_{n-1}$, where $w_{k-1} = (x_0, x_1, \cdots, x_{k-1})$ and $w_{n-1} = (x_k, x_{k+1}, \cdots, x_{n-1})$. Suppose that $w_{k-1} \sim \mathbf{u}_0$ and $w_{n-1} \sim \mathbf{u}_1$ are the equivalences obtained by using the induction hypothesis for w_{k-1} and w_{n-1}. If $p \geqslant |\mathbf{u}_0|, |\mathbf{u}_1|$, then $w \sim y_p(\mathbf{u}_0, \mathbf{u}_1)$ if x_n equals $\vee$ and $w \sim z_p(\mathbf{u}_0, \mathbf{u}_1)$ if x_n equals $\wedge$.

Suppose x_n equals $\vee$. Using 3.4 to eliminate null equations $a_d - b_{p+d} - b_{3p+d}$ for $m < d \leqslant p$ and then renumbering, we can show that $y_p(\mathbf{u}_0, \mathbf{u}_1) \sim$

(W_1, λ_1) as given below:

$$W_1 = F_m(Q_{2n+3,2n+5}, P^{2n+3}_{2k-1}, P^{2n+5}_{2n-1} | U_0, U_1, \cdots, U_{n-1}),$$
$$\lambda_1 = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)^a | \kappa_0, \kappa_1, \cdots, \kappa_{n-1}, \psi_\omega, (\kappa_{k-1})^a, (\kappa_{n-1})^a).$$

If we introduce equations $P^{2n+1}_{2k-1,2n-1}$ by defined variable augmentations at $b_{(2n+1)m+d}$ for $d \leqslant m$, and then apply 3.7 by noting that:

$$[[Q_{2n+3,2n+5}, P^{2n+1}_{2k-1,2n-1}, P^{2n+3}_{2k-1}, P^{2n+5}_{2n-1}]]$$
$$= [[Q_{2n+1}, P^{2n+1}_{2k-1,2n-1}, P^{2n+3}_{2k-1}, P^{2n+5}_{2n-1}]],$$

we obtain $(W_1, \lambda_1) \sim (W_2, \lambda_2)$, where

$$W_2 = F_m(Q_{2n+1}, P^{2n+3}_{2k-1}, P^{2n+5}_{2n-1} | U_0, U_1, \cdots, U_n),$$
$$\lambda_2 = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)^a | \kappa_0, \kappa_1, \cdots, \kappa_n, (\kappa_{k-1})^a, (\kappa_{n-1})^a).$$

(Recall that $U_n \theta_{m,n} = P^{2n+1}_{2k-1,2n-1}$.) But $(W_2, \lambda_2) \sim (J, \gamma_n)$ by eliminating P^{2n+3}_{2k-1} and P^{2n+5}_{2n-1} by defined variable deletions. This proves that $(J, \gamma_n) \sim w(\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_q)$ if x_n equals $\vee$.

Suppose x_n equals $\wedge$. Using 3.4 to eliminate null equations $a_d - b_{p+d}$ and $a_d - b_{3p+d}$ for $m < d \leqslant p$ and renumbering, we can show that $z_p(\mathbf{u}_0, \mathbf{u}_1) \sim (W_3, \lambda_3)$ for

$$W_3 = F_m(Q_{2n+1}, Q_{2n+3}, P^{2n+1}_{2k-1}, P^{2n+3}_{2n-1} | U_0, U_1, \cdots, U_{n-1}),$$
$$\lambda_3 = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)^a | \kappa_0, \kappa_1, \cdots, \kappa_{n-1}, (\kappa_{k-1})^a, (\kappa_{n-1})^a).$$

Since $U_n \theta_{m,n} = P^{2n+1}_{2k-1} \cup P^{2n+1}_{2n-1}$, we use 3.7 and the equation

$$[[Q_{2n+1}, Q_{2n+3}, P^{2n+3}_{2n-1}]] = [[Q_{2n+1}, -Q_{2n+1}, -Q_{2n+3}, P^{2n+1}_{2n-1}]]$$

to obtain $(W_3, \lambda_3) \sim (W_4, \lambda_3)$, where

$$W_4 = F_m(Q_{2n+1}, -Q_{2n+1}, -Q_{2n+3} | U_0, U_1, \cdots, U_n).$$

Finally, $(W_4, \lambda_3) \sim (J, \gamma_n)$ as follows: Make constraint decreases at $b_{(2n+1)m+d}$ and $b_{(2n+3)m+d}$ for $d \leqslant m$ using the equations $-Q_{2n+1}$ and $-Q_{2n+3}$, respectively. Then eliminate $-Q_{2n+1}$ by linear combination deletions and eliminate $-Q_{2n+3}$ by defined variable deletions. This completes the induction proving that $(J, \gamma_n) \sim w(\mathbf{s}_1, \mathbf{s}_2, \cdots \mathbf{s}_q)$, and so we have proved 3.9.

Definition. Let $\mathbf{s} = (G, \alpha)$ and $\mathbf{t} = (H, \beta)$ in $D(K; R)$. Write $\mathbf{s} \ll \mathbf{t}$ if $G \supset H$, $\alpha^a \subset \beta^a$ and $\alpha(b_k) \subset \beta(b_k)$ for all $k \leqslant |\mathbf{t}|$.

3.10. *If* $\mathbf{s} \ll \mathbf{t}$ *in* $D(K; R)$, *then* $\mathbf{t} \vee \mathbf{s} \simeq \mathbf{t}$.

Proof. Assume $\mathbf{s} \ll \mathbf{t}$ in $D(K; R)$ for $\mathbf{s} = (G, \alpha)$ and $\mathbf{t} = (H, \beta)$. Choosing

$m \geqslant |\mathbf{s}|, |\mathbf{t}|$ and using 3.9 and the renumbering function $\mathrm{tr}_{2m}(0, 2)$, we obtain $\mathbf{t} \vee \mathbf{s} \sim \mathbf{F}_m(\mathbf{t}, \mathbf{s}, \vee) \sim (J_1, \gamma_1)$, where

$$J_1 = F_m(Q_{3,5}|\varnothing, G, H) \quad \text{and} \quad \gamma_1 = f_m((\alpha \vee \beta)^a|\psi_\omega, \alpha, \beta).$$

Introducing equations $J = \{b_k - b_{2m+k} - b_{4m+k}\colon k \leqslant |\mathbf{t}|\} \subset P^0_{2,4}$ and equations $P^1_{3,5}$ by defined variable augmentations at b_k for $k \leqslant |\mathbf{t}|$ and at b_{m+k} for $k \leqslant m$, we obtain $(J_1, \gamma_1) \sim (J_2, \gamma_2)$ for

$$J_2 = F_m(Q_{3,5}, J, P^1_{3,5}|\varnothing, G, H) \quad \text{and} \quad \gamma_2 = f_m(\beta^a|\beta, \alpha, \beta).$$

(The hypotheses $\alpha^a \subset \beta^a$ and $\alpha(b_k) \subset \beta(b_k)$ for $k \leqslant |\mathbf{t}|$ were used.) From 3.8 and the hypothesis $H \subset G$, we obtain $[[J_2]] = [[F_m(Q_{3,5}, J, P^1_{3,5}|H, G, \varnothing)]]$. Since $-P^1_{3,5} = P^{3,5}_1$ and $[[Q_{3,5}, P^{3,5}_1]] = [[Q_1, P^{3,5}_1]]$, several applications of 3.7 show that $(J_2, \gamma_2) \sim (J_3, \gamma_2)$ for $J_3 = F_m(Q_1, -J, P^{3,5}_1|H, G, \varnothing)$. But $-J$ and $P^{3,5}_1$ can be eliminated by defined variable deletions at b_{4m+k} for $k \leqslant |\mathbf{t}|$ and at b_{5m+k} for $k \leqslant m$, respectively, again using the hypotheses for α and β. So, $(J_3, \gamma_2) \sim (J_4, \gamma_2)$ for $J_4 = F_m(Q_1|H, G)$. Finally, observe that $(J_4, \gamma_2) \sim \mathbf{F}_m(\mathbf{t})$ by the inessential variables deletion of $G\theta_{m,1}$ using $\{b_k\colon 2m < k \leqslant 6m\}$. Therefore, $\mathbf{t} \vee \mathbf{s} \sim \mathbf{t}$ by 3.9 and transitivity, completing the proof of 3.10.

3.11. *$M(K; R)$ is a modular lattice.*

Proof. By 3.6 and 3.3, $M(K; R)$ has well-defined and commutative meet and join operations. Let $\mathbf{s} = (G, \alpha)$, $\mathbf{t} = (H, \beta)$ and $\mathbf{u} = (J, \gamma)$ in $D(K; R)$, and $m = \max\{|\mathbf{s}|, |\mathbf{t}|, |\mathbf{u}|\}$. To prove the associativity of join in $M(K; R)$, we use 3.9 and observe that $\mathbf{F}_m(\mathbf{s}, \mathbf{t}, \mathbf{u}, \vee, \vee) = (G_1, \alpha_1)$ for $G_1 = F_m(Q_{1,3,5}|G, H, J)$ and $\alpha_1 = f_m((\alpha \vee \beta \vee \gamma)^a|\alpha, \beta, \gamma)$, and that $\mathbf{F}_m(\mathbf{s}, \mathbf{t}, \vee, \mathbf{u}, \vee)$ is the renumbering of (G_1, α_1) by $\mathrm{tr}_{2m}(2, 3)$. Replacing $\vee$ by $\wedge$ and $Q_{1,3,5}$ by Q_1, Q_3, Q_5 above, a proof of the associativity of the meet operation in $M(K; R)$ is obtained.

We now prove that $\mathbf{s}_0 \sim \mathbf{s}_1$ if $\mathbf{s}_0 = (\mathbf{s} \vee \mathbf{t}) \wedge (\mathbf{s} \vee \mathbf{u})$ and $\mathbf{s}_1 = \mathbf{s} \vee (\mathbf{t} \wedge (\mathbf{s} \vee \mathbf{u}))$. By 3.9 and the renumbering function $\mathrm{tr}_{2m}(3, 4)$, we have $\mathbf{s}_0 \sim \mathbf{F}_m(\mathbf{s}, \mathbf{t}, \vee, \mathbf{s}, \mathbf{u}, \vee, \wedge) \sim (H_1, \beta_1)$, where

$$H_1 = F_m(Q_{1,3}, Q_{7,9}|G, H, \varnothing, J, G),$$

$$\beta_1 = f_m(((\alpha \vee \beta) \wedge (\alpha \vee \gamma))^a|\alpha, \beta, \psi_\omega, \gamma, \alpha).$$

If we introduce equations $P^{0,4}_8$ and $P^{1,5}_9$ by defined variable augmentations at b_{4m+k} and b_{5m+k} for $k \leqslant m$, and then apply 3.8 and 3.7 using these equations, we obtain $(H_1, \beta_1) \sim (H_2, \beta_2)$, where

$$H_2 = F_m(Q_{1,3}, Q_{7,9}, P^{0,4}_8, P^{1,5}_9|G, H, G, J, \varnothing), \qquad \beta_2 = f_m((\beta_1)^a|\alpha, \beta, \alpha, \gamma, \alpha).$$

We can now use 3.7 to replace $P_8^{0,4}$ by $P_{0,4}^8$ and $P_9^{1,5}$ by $P_{1,5}^9$ (negation) and also replace $Q_{7,9}$ by $Q_{1,5,7}$, since $[[P_{1,5}^9, Q_{7,9}]] = [[P_{1,5}^9, Q_{1,5,7}]]$. But then it is possible to eliminate $P_{0,4}^8$ and $P_{1,5}^9$ by defined variable deletions at b_{8m+k} and b_{9m+k} for $k \leqslant m$, and obtain $(H_2, \beta_2) \sim (H_3, \beta_3)$ as follows:

$$H_3 = F_m(Q_{1,3}, Q_{1,5,7} | G, H, G, J), \qquad \beta_3 = f_m((\beta_1)^a | \alpha, \beta, \alpha, \gamma).$$

We now add equations $P_{5,7}^3 \subset [[Q_{1,3}, Q_{1,5,7}]]$ by linear combination augmentations, and then make constraint decreases at b_{3m+k} for $k \leqslant m$, using these equations. Then $(H_3, \beta_3) \sim (H_4, \beta_4)$, where

$$H_4 = F_m(Q_{1,3}, Q_{1,5,7}, P_{5,7}^3 | G, H, G, J),$$
$$\beta_4 = f_m((\beta_1)^a | \alpha, \beta_0, \alpha, \gamma) \text{ for } (\beta_0)^a = (\beta \wedge (\alpha \vee \gamma))^a \text{ and } (\beta_0)^b = \beta^b.$$

Finally, we make constraint decreases at a_k for $k \leqslant m$ using the equations of $Q_{1,3}$, then make constraint increases at b_{3m+k} for $k \leqslant m$ using $P_{5,7}^3$ equations and reversing the previous constraint decreases at b_{3m+k}, and then eliminate $P_{5,7}^3$ by linear combination deletions. This proves $(H_4, \beta_4) \sim (H_5, \beta_5)$ as given below:

$$H_5 = F_m(Q_{1,3}, Q_{1,5,7} | G, H, G, J), \qquad \beta_5 = f_m((\alpha \vee (\beta \wedge (\alpha \vee \gamma)))^a | \alpha, \beta, \alpha, \gamma).$$

(We have used the fact that $(\beta_1)^a \wedge (\beta_5)^a = (\beta_5)^a$ because $x \vee (y \wedge (x \vee z)) \subset (x \vee y) \wedge (x \vee z)$ in the lattice K.) But $(H_5, \beta_5) = \mathbf{F}_m(\mathbf{s}, \mathbf{t}, \mathbf{s}, \mathbf{u}, \vee, \wedge, \vee) \sim \mathbf{s}_1$ by 3.9, so we have proved the required equivalence $\mathbf{s}_0 \sim \mathbf{s}_1$.

We now prove the absorption laws for $M(K; R)$. It is easily checked that $\mathbf{F}_m(\mathbf{s}, \mathbf{t}, \wedge) \ll \mathbf{F}_m(\mathbf{s})$. But then we obtain the absorption equivalence $\mathbf{s} \vee (\mathbf{s} \wedge \mathbf{t}) \sim \mathbf{s}$ by 3.10, 3.6 and 3.9. Furthermore, if we substitute $\mathbf{s} \wedge \mathbf{t}$ for $\mathbf{u}$ in the equivalence $\mathbf{s}_0 \sim \mathbf{s}_1$ above, and then reduce both sides of the resulting equivalence using the above absorption equivalence, 3.3 and 3.6, we obtain a proof of the dual absorption equivalence $\mathbf{s} \wedge (\mathbf{s} \vee \mathbf{t}) \sim \mathbf{s}$. Therefore, $M(K; R)$ is a lattice. But then the equivalence $\mathbf{s}_0 \sim \mathbf{s}_1$ proved above implies that $M(K; R)$ is modular, completing the proof of 3.11.

3.12. *For any* (0, 1) *lattice* K, ψ: $K \to M(K; R)$ *is a lattice homomorphism such that* $\psi(\omega)$ *is a minimum element for* $M(K; R)$.

PROOF. For x, y in K, we can show that $(\varnothing, \psi_{x \vee y})$ is a union deletion of $(\varnothing, \psi_x) \vee (\varnothing, \psi_y)$, and so they are equivalent. Starting from $(\varnothing, \psi_x) \wedge (\varnothing, \psi_y)$, replace $a_1 - b_j$ by $-a_1 + b_j$ for $j = 2, 4$ using 3.7. Then constraint decreases and defined variable deletions at b_2 and b_4 lead to the result $(\varnothing, \psi_{x \wedge y}) \sim (\varnothing, \psi_x) \wedge (\varnothing, \psi_y)$. Therefore, ψ is a lattice homomorphism. If $s \wedge (\varnothing, \psi_\omega) = (J, \gamma)$ in $D(K; R)$, we observe that $\gamma^a = \psi_\omega$. Therefore, $s \wedge (\varnothing, \psi_\omega) \sim (\varnothing, \psi_\omega)$ by 3.4, and so $\psi(\omega)$ is a minimum element for $M(K; R)$. This proves 3.12.

DEFINITION. Let $m \geqslant 1$. For g in $F_R(V)$, $\lambda_m(g)$ denotes the h in $F_R(V)$ such that $h(a_k) = 0$ for $k \leqslant m$, $h(a_k) = g(a_{k-m})$ for $k > m$, and $h(b_k) = g(b_k)$ for $k \geqslant 1$. If $G \subset F_R(V)$, let $\lambda_m(G)$ denote $\{\lambda_m(g): g \in G\}$. Similarly, if α is a lattice constraint function, let $\lambda_m(\alpha)$ denote the lattice constraint function β such that $\beta(a_k) = \omega$ if $k \leqslant m$, $\beta(a_k) = \alpha(a_{k-m})$ if $k > m$, and $\beta(b_k) = \alpha(b_k)$ for $k \geqslant 1$.

Let $D_m(r_0, r_1, \cdots, r_n)$ for $r_0, r_1, \cdots, r_n$ in R denote

$$\{r_0 a_k + r_1 a_{m+k} + r_2 a_{2m+k} + \cdots + r_n a_{nm+k}: k = 1, 2, \cdots, m\}.$$

For example, $D_m(1, r) = \{a_k + r a_{m+k}: k \leqslant m\}$.

If $\mathrm{s} = (G, \alpha)$ in $D(K; R)$, let $\pi_m(\mathrm{s})$ denote $(\lambda_m(G), \lambda_m(\alpha))$ in $D(K; R)$, and let $\delta_m(\mathrm{s}, r)$ denote $(\lambda_m(G) \cup D_m(1, r), \delta_m(\alpha))$ in $D(K; R)$ for r in R, where $\delta_m(\alpha)^b = \alpha^b$, $\delta_m(\alpha)(a_k) = \alpha(a_k)$ for $k \leqslant m$, and $\delta_m(\alpha)(a_k) = \alpha(a_{k-m})$ for $k > m$. If $m \geqslant |\mathrm{s}|$, then $\delta_m(\alpha) = \alpha^a \vee \lambda_m(\alpha)$.

If $\mathrm{s} \sim \mathrm{t}$ in $D(K; R)$, then $\pi_m(\mathrm{s}) \sim \pi_m(\mathrm{t})$ and $\delta_m(\mathrm{s}, r) \sim \delta_m(\mathrm{t}, r)$ for r in R can be verified without much difficulty. If $x = [\mathrm{s}]$ in $M(K; R)$, therefore, we can define $\pi_m(x) = [\pi_m(\mathrm{s})]$ and $\delta_m(x, r) = [\delta_m(\mathrm{s}, r)]$ in $M(K; R)$, independently of the choice of representative s of x.

Let $|x|$ denote $\min\{|\mathrm{s}|: x = [\mathrm{s}]\}$ for x in $M(K; R)$. We will consider $\pi_m(x)$ and $\delta_m(x, r)$ only when $m \geqslant |x|$. Observe that $|\pi_m(x)| \leqslant m + |x|$. Furthermore, $|\delta_m(x, r)| \leqslant m + |x|$, deleting null equations from $D_m(1, r)$ by 3.4 if necessary.

Interpretation. Given an embedding $\iota: K \to \Gamma(M; R)$ such that $\iota(\omega) = 0$, let the solution set $\nu^*(x)$ of x in $M(K; R)$ be the common solution set $\nu(G, \alpha, \iota)$ of all the representatives (G, α) of x, (G, α) in $D(K; R)$. For $k = |x|$, the solutions of x in $\Gamma_f(M^{\mathrm{N}}; R)$ are certain sequences of the form $(v_1, v_2, \cdots, v_k, 0, 0, \cdots)$, where $v_i \in M$ for $i \leqslant k$. If $m \geqslant |x|$, then $(v_1, v_2, \cdots, v_k, 0, 0, \cdots)$ is in $\nu^*(x)$ if and only if the two sequences below are in $\nu^*(\pi_m(x))$ and $\nu^*(\delta_m(x, r))$, respectively:

$$(0, 0, \cdots, 0, v_1, v_2, \cdots, v_k, 0, 0, \cdots),$$

$$(-rv_1, -rv_2, \cdots, -rv_k, 0, \cdots, 0, v_1, v_2, \cdots, v_k, 0, 0, \cdots).$$

(In both cases above, v_i appears in the $(m + i)$th position of the sequence.) That is, $\nu^*(\pi_m(x))$ is an R-module which is isomorphic to $\nu^*(x)$ and is disjoint from $\nu^*(x)$ in $\Gamma_f(M^{\mathrm{N}}; R)$, obtained by translating m coordinate positions to the right. Furthermore, $\nu^*(\delta_m(x, r))$ is the "negative graph" (see [7, §2]) of the R-homomorphism rF, where F: $\nu^*(\pi_m(x)) \to \nu^*(x)$ is the translation isomorphism. (Since R is commutative, rF is R-linear.) In particular, $\nu^*(\delta_m(x, 1))$ is the negative graph of F and of its reciprocal F^{-1}: $\nu^*(x) \to \nu^*(\pi_m(x))$. In $\mathrm{A}_{M(K;R)}$, maps can be specified by suitable pairs of lattice elements, each element acting like the negative graph of an R-homomorphism. To specify $\zeta_A(r)$: $A \to A$ for $A = x/\psi(\omega)$

in $\mathbf{A}_{M(K;R)}$, we will use the elements $\delta_m(x, r)$ and $\delta_m(x, 1)$ for $m = |x|$. This corresponds to the composite

$$\nu^*(x) \xrightarrow[F^{-1}]{} \nu^*(\pi_m(x)) \xrightarrow[rF]{} \nu^*(x)$$

in the interpretation. The definition of $\zeta_A(r)$ for arbitrary objects A of $\mathbf{A}_{M(K;R)}$ is a slight modification of the definition described for the case above.

3.13. *If x is in $M(K; R)$, r is in R and $m \geqslant |x|$, then $\delta_m(x, r) \subset x \vee \pi_m(x)$, $\pi_m(x) \subset x \vee \delta_m(x, r)$ and $r = 1$ implies that $x \subset \pi_m(x) \vee \delta_m(x, r)$. Furthermore, $x \wedge \delta_m(x, r) = \psi(\omega)$, and $r = 1$ implies that $\pi_m(x) \wedge \delta_m(x, r) = \psi(\omega)$. If $x_0, x_1, \cdots, x_d$ are in $M(K; R)$ and $n \geqslant |x_1|, |x_2|, \cdots, |x_d|$, then $\pi_n(x_0) \wedge (x_1 \vee x_2 \vee \cdots \vee x_d) = \psi(\omega)$. If $y \subset z$ in $M(K; R)$ and $p \geqslant 1$, then $\pi_p(y) \subset \pi_p(z)$ and $\delta_p(y, r) \subset \delta_p(z, r)$.*

PROOF. Assume the hypotheses for x, r and m, and choose $\mathrm{s} = (G, \alpha)$ in $D(K; R)$ such that $x = [\mathrm{s}]$ and $m \geqslant |\mathrm{s}|$. Using 3.9 and the renumbering $\mathrm{tr}_{4m}(0, 2)$, we have $\delta_m(\mathrm{s}, r) \sim \mathbf{F}_{2m}(\delta_m(\mathrm{s}, r)) \sim (J_1, \gamma_1)$, where

$$J_1 = F_{2m}(Q_5(2m)|\varnothing, \varnothing, D_m(1, r) \cup \lambda_m(G)),$$
$$\gamma_1 = f_{2m}((\alpha^a \vee \lambda_m(\alpha))^a|\psi_\omega, \psi_\omega, \alpha^a \vee \lambda_m(\alpha)).$$

By defined variable augmentations, the following sets of equations are introduced: $P_{10}^2(m)$, $P_8^4(m)$, $P_{11}^7(m)$ and $J' = \{b_k + rb_{8m+k}: k \leqslant m\}$. So, $(J_1, \gamma_1) \sim (J_2, \gamma_2)$ as given below.

$$J_2 = F_{2m}(Q_5(2m), P_{10}^2(m), P_8^4(m), P_{11}^7(m), J'|\varnothing, \varnothing, D_m(1, r) \cup \lambda_m(G)),$$
$$\gamma_2 = f_{2m}((\alpha^a \vee \lambda_m(\alpha))^a|\alpha, \lambda_m(\alpha), \alpha^a \vee \lambda_m(\alpha)).$$

Now $Q_2(m) \subset [[Q_5(2m), P_{10}^2(m)]]$ and $\gamma_2(b_{6m+k}) = \lambda_m(\alpha)(a_k) = \omega$ for $k \leqslant m$, so $(J_2, \gamma_2) \sim (J_2 \cup Q_{2,6}(m), \gamma_2)$ by linear combination augmentations followed by null variable augmentations. Similarly, we can add the remaining m equations of $Q_{1,3}(2m)$ not in $Q_{2,6}(m)$ by linear combination augmentations from $[[Q_5(2m), P_{11}^7(m)]]$ followed by null variable augmentations. Furthermore, we have

$$G\theta_{2m,0} \subset [[\lambda_m(G)\theta_{2m,2}, J', P_{10}^2(m), D_m(1, r)\theta_{2m,2}]] \quad \text{and}$$
$$\lambda_m(G)\theta_{2m,1} \subset [[\lambda_m(G)\theta_{2m,2}, P_8^4(m), P_{11}^7(m)]].$$

The first inclusion follows because g in G implies $g\theta_{2m,0}$ equals

$$-r\lambda_m(g)\theta_{2m,2} + \sum_{k=1}^{m} [g(b_k)(b_k + rb_{8m+k}) + g(a_k)(b_{2m+k} + rb_{11m+k})],$$

and $b_{2m+k} + rb_{11m+k}$ is the sum of $b_{2m+k} - b_{10m+k}$ in $P^2_{10}(m)$ and $b_{10m+k} + rb_{11m+k}$ in $D_m(1, r)\theta_{2m,2}$. The second inclusion follows from 3.8, so $(J_2, \gamma_2) \sim (J_3, \gamma_2)$ for $J_3 = J_2 \cup F_{2m}(Q_{1,3}(2m)|G, \lambda_m(G))$. But

$$(J_3, \gamma_2) \ll \mathrm{F}_{2m}(\mathrm{s}, \pi_m(\mathrm{s}), \vee),$$

and so $\delta_m(x, r) \subset x \vee \pi_m(x)$ by 3.9 and 3.10.

By arguments similar to the foregoing, we obtain $\pi_m(\mathrm{s}) \sim (H_1, \beta_1) \sim (H_2, \beta_2)$, given as follows:

$$H_1 = F_{2m}(Q_5(2m)|\varnothing, \varnothing, \lambda_m(G)),$$

$$\beta_1 = f_{2m}(\lambda_m(\alpha)^a|\psi_\omega, \psi_\omega, \lambda_m(\alpha)),$$

$$H_2 = F_{2m}(Q_5(2m), P^4_8(m), P^7_{11}(m), H', P^{2,6}(m)|\varnothing, D_m(1, r), \lambda_m(G)),$$

$$\beta_2 = f_{2m}(\lambda_m(\alpha)^a|\alpha, \alpha^a \vee \lambda_m(\alpha), \lambda_m(\alpha)),$$

$$H' = \{b_k - rb_{8m+k}\colon k \leqslant m\}.$$

(Note that $D_m(1, r)\theta_{2m,1}$ is obtained by defined variable augmentations at b_{6m+k} for $k \leqslant m$, and that these equations must be introduced before those in $P^{2,6}(m)$.) We can obtain equations $Q_{2,6}(m)$ by negating $P^{2,6}(m)$ using 3.7 and then making null variable deletions and then augmentations at each a_k, $k \leqslant m$. The m equations of $Q_{1,3}(2m) - Q_{2,6}(m)$ can be obtained by the method previously used. We see that

$$G\theta_{2m,0} \subset [[\lambda_m(G)\theta_{2m,2}, H', D_m(1, r)\theta_{2m,1}, P^{2,6}(m), P^7_{11}(m)]],$$

observing that $b_{2m+k} - rb_{11m+k}$ is in $[[D_m(1, r)\theta_{2m,1}, P^{2,6}(m), P^7_{11}(m)]]$ for all $k \leqslant m$. Using the previous inclusion for $\lambda_m(G)\theta_{2m,1}$, we obtain $(H_2, \beta_2) \sim (H_3, \beta_2)$ for $H_3 = H_2 \cup F_{2m}(Q_{1,3}(2m)|G, \lambda_m(G))$. But then $(H_3, \gamma_2) \ll \mathrm{F}_{2m}(\mathrm{s}, \delta_m(\mathrm{s}, r), \vee)$ implies $\pi_m(x) \subset x \vee \delta_m(x, r)$ by 3.9, 3.10. We will omit the similar proof that $r = 1$ implies $x \subset \pi_m(x) \vee \delta_m(x, r)$. (The condition $r = 1$ is needed so that the equations $b_{6m+k} + rb_{7m+k}$ of $D_m(1, r)\theta_{2m,1}$ can be introduced by defined variable augmentations at b_{7m+k}.) This completes the proof of the first part.

By 3.9 and 3.8, $\mathrm{s} \wedge \delta_m(\mathrm{s}, r) \sim \mathrm{F}_{2m}(\delta_m(\mathrm{s}, r), \mathrm{s}, \wedge) \sim (H, \beta)$, where $H = F_{2m}(Q_1, Q_3, D_m(1, r)|\lambda_m(G), G)$ and $\beta = f_{2m}(\alpha^a|\alpha^a \vee \lambda_m(\alpha), \alpha)$. But $(H, \beta) \sim (H, \beta_0)$ for $\beta_0^a = \psi_\omega$ and $\beta_0^b = \beta^b$ by constraint decreases at a_k using $a_k + ra_{m+k}$ for each $k \leqslant m$. Since $(H, \beta_0) \sim (\varnothing, \psi_\omega)$ by 3.4, we have $x \wedge \delta_m(x, r) = \psi(\omega)$. The proof that $r = 1$ implies that $\pi_m(x) \wedge \delta_m(x, r) = \psi(\omega)$ is similar, completing the second part.

Assume the hypotheses for $x_0, x_1, \cdots, x_d$ and n. For each i, $0 \leqslant i \leqslant d$, choose $\mathrm{s}_i = (G_i, \alpha_i)$ such that $x_i = [\mathrm{s}_i]$ and $n \geqslant |\mathrm{s}_i|$ if $i \geqslant 1$. Using 3.9, $\pi_n(x_0) \wedge$

$(x_1 \vee x_2 \vee \cdots \vee x_d) = [J, \gamma]$ for some (J, γ) in $D(K; R)$ such that $\gamma^a = (\lambda_n(\alpha_0) \wedge (\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_d))^a$. But $\gamma^a = \psi_\omega$ is easily verified, and so $[J, \gamma] = [\varnothing, \psi_\omega] = \psi(\omega)$ by 3.4, proving the third part.

Suppose $y \subset z$ and $p \geqslant 1$. Choose s_0 and t_0 such that $y = [\mathrm{s}_0]$ and $z = [\mathrm{t}_0]$, and let $q \geqslant |\mathrm{s}_0|, p + |\mathrm{t}_0|$. Then $y = [\mathrm{s}]$ and $z = [\mathrm{t}]$ for $\mathrm{s} = \mathrm{F}_q(\mathrm{t}_0, \mathrm{s}_0, \wedge)$ and $\mathrm{t} = \mathrm{F}_q(\mathrm{t}_0)$, using 3.9. By inspection, $\mathrm{s} \ll \mathrm{t}$, $\pi_p(\mathrm{s}) \ll \pi_p(\mathrm{t})$ and $\delta_p(\mathrm{s}, r) \ll \delta_p(\mathrm{t}, r)$. But then $\pi_p(y) \subset \pi_p(z)$ and $\delta_p(y, r) \subset \delta_p(z, r)$ by 3.10, completing the proof of 3.13.

3.14. *$M(K; R)$ is an abelian lattice.*

PROOF. By 3.11 and 3.12, $M(K; R)$ is a modular lattice with smallest element $\psi(\omega)$. To prove that every x in $M(K; R)$ can be "tripled", we observe by 3.13 that $m \geqslant |x|$ implies

$$x \vee \pi_m(x) = x \vee \delta_m(x, 1) = \pi_m(x) \vee \delta_m(x, 1), \quad \text{and}$$

$$x \wedge \pi_m(x) = x \wedge \delta_m(x, 1) = \pi_m(x) \wedge \delta_m(x, 1) = \psi(\omega).$$

Therefore, $M(K; R)$ is an abelian lattice by [7, 4.1, p. 181]. (Note that the above proof simply adapts the proof of [7, 4.2].)

We next prove two complex inclusion relations in $M(K; R)$ that are needed to complete the verification of 2.6.

3.15. *Suppose x, y, z are in $M(K; R)$, $m \geqslant |x|, |y|$, $p \geqslant 2m$ and $z \subset x \vee \pi_m(y)$. For r in R, let $w(r)$ denote $(\delta_p(x, r) \vee z) \wedge (\pi_p(x) \vee \pi_m(y))$ in $M(K; R)$. Then for any r_1, r_2 in R, it follows that*

$$(w(r_2) \vee \delta_m(y, r_1)) \wedge (\pi_p(x) \vee y) \subset (w(r_1 r_2) \vee \delta_m(y, 1)) \wedge (\pi_p(x) \vee y).$$

PROOF. Assuming the hypotheses, choose $\mathrm{s} = (G, \alpha)$, $\mathrm{t} = (H, \beta)$ and u_0 in $D(K; R)$ such that $x = [\mathrm{s}]$, $y = [\mathrm{t}]$, $z = [\mathrm{u}_0]$ and $m \geqslant |\mathrm{s}|, |\mathrm{t}|$. Because of the hypothesis $z \subset x \vee \pi_m(y)$, we have by 3.9 that, for a sufficiently large j, $\mathrm{u}_0 \sim \mathrm{u} = (J, \gamma) = \mathrm{F}_j(\mathrm{u}_0, \mathrm{s}, \pi_m(\mathrm{t}), \vee, \wedge)$. Note that $\gamma^a \subset (\alpha \vee \lambda_m(\beta))^a$, and so $\gamma(a_i) = \omega$ if $i > 2m$.

Suppose $k \geqslant 2p, |\mathrm{u}|$ and $n = 8k$, and let $D_0(r)$ denote $\{ra_{p+i} - b_{11k+i}: i \leqslant m\}$. We intend to define G_0, H_0 and α_0 independently of r such that $w(r) = [\mathrm{w}(r)]$ for $\mathrm{w}(r) = (D_0(r) \cup G_0 \cup H_0, \alpha_0)$, g in G_0 implies $g(a_i) = 0$ unless $p < i \leqslant p + m$ and $g(b_i) = 0$ unless $i \leqslant n$, h in H_0 implies $h(a_i) = 0$ unless $m < i \leqslant 2m$ and $h(b_i) = 0$ unless $n < i \leqslant 2n$, $\alpha_0^a \subset (\lambda_p(\alpha) \vee \lambda_m(\beta))^a$ and $\alpha_0(b_i) = \omega$ for $i > 2n$. Now $w(r) = [\mathrm{t}_1(r)]$ for $\mathrm{t}_1(r) = \mathrm{F}_k(\delta_p(\mathrm{s}, r), \mathrm{u}, \vee, \pi_p(\mathrm{s}), \pi_m(\mathrm{t}), \vee, \wedge)$ by 3.9. Using the renumbering $\mathrm{tr}_{2k}(1, 5)$, we have $\mathrm{t}_1(r) \sim (H_1, \alpha_0)$, as given below.

$$H_1 = F_k(Q_{1,11}, Q_{7,9} | D_p(1, r) \cup \lambda_p(G), \emptyset, \emptyset, \lambda_p(G), \lambda_m(H), J),$$
$$\alpha_0^a = ((\alpha^a \vee \lambda_p(\alpha) \vee \gamma) \wedge (\lambda_p(\alpha) \vee \lambda_m(\beta))^a,$$
$$\alpha_0 = f_k(\alpha_0^a | \alpha^a \vee \lambda_p(\alpha), \psi_\omega, \psi_\omega, \lambda_p(\alpha), \lambda_m(\beta), \gamma).$$

Note that $\alpha_0^a \subset (\lambda_p(\alpha) \vee \lambda_m(\beta))^a$ and $\alpha_0(b_i) = \omega$ unless $i \leqslant 12k < 2n$. Using 3.4, replace $D_p(1, r)\theta_{k,0}$ by $D_1(r) = \{b_{k+i} + b_{k+p+i}\colon i \leqslant m\}$, by deleting $p - m$ null equations. Then $D_1(r)$ can be replaced by

$$D_2(r) = \{a_i - b_{11k+i} + ra_{p+i} - rb_{11k+p+i}\colon i \leqslant m\}$$

by 3.7, since $[[D_1(r), Q_{1,11}(k)]] = [[D_2(r), Q_{1,11}(k)]]$. But $\alpha_0(a_i) = \alpha_0(b_{11k+p+i})$ $= \omega$ for $i \leqslant m$, so $D_2(r)$ can be replaced by $D_0(r)$ by 3.4. So, we obtain (H_1, α_0) $\sim (H_2, \alpha_0)$, where

$$H_2 = F_k(D_0(r), Q_{1,11}, Q_{7,9} | \lambda_p(G), \emptyset, \emptyset, \lambda_p(G), \lambda_m(H), J).$$

We now define $H_3 = F_k(D_0(r), H_4, H_5, H_6, H_7, H_8 | \lambda_p(G), \emptyset, \emptyset, \lambda_p(G), \lambda_m(H), J)$, where

$$H_4 = \{-b_{k+i} - b_{11k+i}\colon i \leqslant m\}, \qquad H_5 = \{a_{m+i} - b_{11k+m+i}\colon i \leqslant m\},$$
$$H_6 = \{a_{p+i} - b_{k+p+i}\colon i \leqslant m\}, \qquad H_7 = \{a_{m+i} - b_{9k+m+i}\colon i \leqslant m\} \quad \text{and}$$
$$H_8 = \{a_{p+i} - b_{7k+p+i}\colon i \leqslant m\}.$$

By 3.4, we can replace $Q_{1,11}(k)$ by $H_4 \cup H_5 \cup H_6$ and $Q_{7,9}(k)$ by $H_7 \cup H_8$, proving $(H_2, \alpha_0) \sim (H_3, \alpha_0)$. Furthermore, the equations of H_4 can be eliminated by first replacing H_4 by $-H_4$ by 3.7, and then using constraint decreases and defined variable deletions of $b_{k+i} + b_{11k+i}$ at b_{k+i}, since $\gamma^a \subset (\alpha \vee \lambda_m(\beta))^a$. So, if we define

$$G_0 = F_k(H_6, H_8 | \lambda_p(G), \emptyset, \emptyset, \lambda_p(G)) \quad \text{and}$$
$$H_0 = F_k(H_5, H_7 | \emptyset, \emptyset, \emptyset, \emptyset, \lambda_m(H), J),$$

we obtain $(H_3, \alpha_0) \sim (D_0(r) \cup G_0 \cup H_0, \alpha_0)$. This proves $w(r) = [\mathrm{w}(r)]$ for $\mathrm{w}(r) = (D_0(r) \cup G_0 \cup H_0, \alpha_0)$, and it is easily seen that G_0, H_0 and α_0 have the required properties.

The inclusion relation we must prove follows from 3.9 and 3.10 if we can construct (J_0, γ_0) in $D(K; R)$ such that

$$\mathbf{u}^*(r_1, r_2) = \mathbf{F}_{2n}(\mathrm{w}(r_2), \delta_m(\mathrm{t}, r_1), \surd, \pi_p(\mathrm{s}), \mathrm{t}, \vee, \wedge) \sim (J_0, \gamma_0), \quad \text{and}$$
$$(J_0, \gamma_0) \ll \mathbf{u}^*(r_1 r_2, 1) = \mathbf{F}_{2n}(\mathrm{w}(r_1 r_2), \delta_m(\mathrm{t}, 1), \vee, \pi_p(\mathrm{s}), \mathrm{t}, \vee, \wedge).$$

First, we use the renumbering $\mathrm{tr}_{4n}(0, 5)\mathrm{tr}_{4n}(1, 6)$ to establish $\mathbf{u}^*(r_1, r_2) \sim (J_1, \gamma_1)$, for J_1 and γ_1 as follows, respectively:

$$F_{2n}(Q_{7,9}, Q_{11,13} | \varnothing, \varnothing, \varnothing, \lambda_p(G), H, D_0(r_2) \cup G_0 \cup H_0, D_m(1, r_1) \cup \lambda_m(H)),$$

$$f_{2n}(\gamma_1^a | \psi_\omega, \psi_\omega, \psi_\omega, \lambda_p(\alpha), \beta, \alpha_0, \beta^a \vee \lambda_m(\beta)),$$

$$\text{where } \gamma_1^a = ((\alpha_0 \vee \beta^a \vee \lambda_m(\beta)) \wedge (\lambda_p(\alpha) \vee \beta))^a.$$

We now introduce by defined variable augmentations the sets of equations $G_1, G_2, \cdots, G_7$, where

$$G_1 = P_{20}^0(n), \qquad G_2 = \{b_{n+i} - r_1 b_{21n+i}\colon i \leqslant n\},$$

$$G_3 = \{b_{2n+m+i} - r_1 b_{22n+m+i}\colon i \leqslant m\},$$

$$G_4 = \{b_{2n+p+i} - b_{22n+p+i}\colon i \leqslant m\},$$

$$G_5 = \{b_{4n+i} - r_1 b_{24n+i}\colon i \leqslant n\}, \qquad G_6 = \{b_{6n+i} - b_{26n+i}\colon i \leqslant m\} \text{ and}$$

$$G_7 = \{b_{6n+m+i} - r_1 b_{26n+m+i}\colon i \leqslant m\}.$$

Then we obtain $(J_1, \gamma_1) \sim (J_2, \gamma_2)$, where $J_2 = J_1 \cup G_1 \cup G_2 \cup \cdots \cup G_7$ and

$$\gamma_2 = f_{2n}(\gamma_1^a | \alpha_0, \beta^a \vee \lambda_m(\beta), \psi_\omega, \lambda_p(\alpha), \beta, \alpha_0, \beta^a \vee \lambda_m(\beta)).$$

By 3.8 and the properties of G_0, H_0 and $\lambda_m(H)$, we see that

$$G_0\theta_{2n,0} \subset [[G_0\theta_{2n,5}, G_1, G_4]], \quad H_0\theta_{2n,0} \subset [[H_0\theta_{2n,5}, G_2, G_3]] \quad \text{and}$$

$$\lambda_m(H)\theta_{2n,1} \subset [[\lambda_m(H)\theta_{2n,6}, G_5, G_7]].$$

Furthermore, $D_0(r_1 r_2)\theta_{2n,0} \subset [[D_0(r_2)\theta_{2n,5}, G_2, G_4]]$ and $D_m(1, 1)\theta_{2n,1} \subset [[D_m(1, r_1)\theta_{2n,6}, G_6, G_7]]$ are easily verified. Therefore, $(J_2, \gamma_2) \sim (J_3, \gamma_2)$, where

$$J_3 = J_2 \cup F_{2n}(\varnothing | D_0(r_1 r_2) \cup G_0 \cup H_0, D_m(1, 1) \cup \lambda_m(H)).$$

We define $J_0 = J_3 \cup Q_{1,3}(2n)$ and $\gamma_0 = \gamma_2$, and observe that $(J_0, \gamma_0) \ll \mathbf{u}^*(r_1 r_2, 1)$. Now the equations $a_i - b_{6n+i} - b_{22n+i}$ are in $[[Q_{11,13}(2n), G_6]]$, and $\gamma_2(b_{2n+i}) = \gamma_2(b_{22n+i}) = \omega$, for each $i \leqslant m$. So, $(J_3, \gamma_2) \sim (J_4, \gamma_2)$ such that J_4 contains J_3 and the first m equations of $Q_{1,3}(2n)$, by 3.4. We can add the equations $a_i - b_{2n+i} - b_{6n+i}$ to (J_4, γ_2) for $m < i \leqslant 2m$ and for $p < i \leqslant p + m$ by similar arguments. The remaining equations of $Q_{1,3}(2n)$ are null equations for γ_2, and so can also be added to (J_4, γ_2) by 3.4, giving the result $(J_4, \gamma_2) \sim (J_0, \gamma_0)$. So, $\mathbf{u}^*(r_1, r_2) \sim (J_0, \gamma_0) \ll \mathbf{u}^*(r_1 r_2, 1)$, completing the proof of 3.15.

3.16. *Suppose* $x \in M(K; R)$ *and* $m \geqslant |x|$. *Let*

$$y_1 = (\delta_m(x, 1) \vee \delta_{2m}(x, 1)) \wedge (\pi_m(x) \vee \pi_{2m}(x)),$$

$$y_2 = (\delta_m(x, 1) \vee \delta_{3m}(x, 1)) \wedge (\pi_m(x) \vee \pi_{3m}(x)) \quad \textit{and}$$

$$y_3 = (y_1 \vee \pi_{3m}(x)) \wedge (y_2 \vee \pi_{2m}(x)).$$

Then for any $r_1, r_2 \in R$, *it follows that*

$$(y_3 \vee \delta_{2m}(x, r_1) \vee \delta_{3m}(x, r_2)) \wedge (x \vee \pi_m(x)) \subset \delta_m(x, r_1 + r_2).$$

PROOF. Assume the hypotheses. Choose $\mathrm{s} = (G, \alpha)$ in $D(K; R)$ such that $x = [\mathrm{s}]$ and $m \geqslant |\mathrm{s}|$. For $p = 4m$, we have by 3.9 that $y_1 = [J_1, \gamma_1]$, where J_1 and γ_1 equal, respectively,

$$F_p(Q_{1,3}, Q_{7,9} | D_m(1, 1) \cup \lambda_m(G), D_{2m}(1, 1) \cup \lambda_{2m}(G), \varnothing, \lambda_m(G), \lambda_{2m}(G)),$$

$$f_p((\lambda_m(\alpha) \vee \lambda_{2m}(\alpha))^a | \alpha^a \vee \lambda_m(\alpha), \alpha^a \vee \lambda_{2m}(\alpha), \psi_\omega, \lambda_m(\alpha), \lambda_{2m}(\alpha)).$$

For $k \leqslant m$, observe that the equation $a_k + a_{m+k} + a_{2m+k} - b_{p+2m+k} - b_{3p+m+k}$ is in $[[Q_{1,3}, D_m(1, 1)\theta_{p,0}, D_{2m}(1, 1)\theta_{p,1}]]$. But $\gamma_1(a_k) = \gamma_1(b_{p+2m+k}) = \gamma_1(b_{3p+m+k}) = \omega$ for $k \leqslant m$, so by 3.7 and 3.4 we have $y_1 = [\mathrm{t}_1]$, where $\mathrm{t}_1 = (D_m(0, 1, 1) \cup J_1, \gamma_1)$. By a similar argument, one can show that $y_2 = [\mathrm{t}_2]$, where $\mathrm{t}_2 = (D_m(0, 1, 0, 1) \cup J_2, \gamma_2)$ for a suitable J_2 and γ_2 such that $\gamma_2^a = (\lambda_m(\alpha) \vee \lambda_{3m}(\alpha))^a$.

Choosing $n \geqslant 6m, |\mathrm{t}_1|, |\mathrm{t}_2|$, we observe by 3.12 and 3.9 that

$$(y_3 \vee \delta_{2m}(x, r_1) \vee \delta_{3m}(x, r_2)) \wedge (\pi_m(x) \vee x) = [\mathrm{F}_n(x_0, x_1, \cdots, x_{14})],$$

where $(x_0, x_1, \cdots, x_{14})$ is the $D(K; R)$ polynomial:

$$(\pi_m(\mathrm{s}), \mathrm{s}, \vee, \mathrm{t}_1, \pi_{3m}(\mathrm{s}), \vee, \mathrm{t}_2, \pi_{2m}(\mathrm{s}), \vee, \wedge, \delta_{2m}(\mathrm{s}, r_1), \vee, \delta_{3m}(\mathrm{s}, r_2), \vee, \wedge).$$

Let $\mathrm{F}_n(x_0, x_1, \cdots, x_{14}) = (J_3, \gamma_3)$. Then

$$J_3 = F_n(Q(I_{14}) | G_0, G_1, \cdots, G_{14}),$$

where $G_0 = \lambda_m(G)$, $D_m(0, 1, 1) \subset G_3$, $D_m(0, 1, 0, 1) \subset G_6$, $D_m(1, 0, r_1) \subset D_{2m}(1, r_1) \subset G_{10}$, $D_m(1, 0, 0, r_2) \subset D_{3m}(1, r_2) \subset G_{12}$, and $Q(I_{14}) = Q_{1,3} \cup Q_{7,9,21,25} \cup Q_{13,15,21,25}$. Furthermore,

$$\gamma_3 = f_n((\lambda_m(\alpha) \vee \alpha)^a | \beta_0, \beta_1, \cdots, \beta_{14}),$$

where $\beta_0 = \alpha^a \vee \lambda_m(\alpha)$, $\beta_1 = \alpha$, $\beta_3^a = (\lambda_m(\alpha) \vee \lambda_{2m}(\alpha))^a$, $\beta_4 = \lambda_{3m}(\alpha)$, $\beta_7 = \lambda_{2m}(\alpha)$, $\beta_{10} = \alpha^a \vee \lambda_{2m}(\alpha)$ and $\beta_{12} = \alpha^a \vee \lambda_{3m}(\alpha)$.

For g in G, the equation $\lambda_m(g) - \Sigma_{k=1}^m g(a_k) b_{3n+m+k}$ is in $[[Q_{1,3}(n), \lambda_m(G)\theta_{n,0}]]$, and $\gamma_3(b_{3n+m+k}) = \beta_1(a_{m+k}) = \omega$ for all $k \leqslant m$. Similarly, the equation

$$a_k + (r_1 + r_2)a_{m+k} + r_1 a_{2m+k} + r_2 a_{3m+k} - b_{7n+k} - b_{9n+k} - r_1 b_{9n+m+k}$$
$$- r_2 b_{15n+m+k} - (r_1 + r_2)b_{21n+m+k} - (r_1 + r_2)b_{25n+m+k}$$
$$- r_1 b_{9n+2m+k} - r_1 b_{25n+2m+k} - r_2 b_{15n+3m+k} - r_2 b_{21n+3m+k}$$

is in $[[Q_{7,9,21,25}, Q_{13,15,21,25}, G_3\theta_{n,3}, G_6\theta_{n,6}, G_{10}\theta_{n,10}, G_{12}\theta_{n,12}]]$ for each $k \leqslant m$, using the properties given for G_3, G_6, G_{10} and G_{12} above. But

$$\begin{aligned}\gamma_3(a_{2m+k}) &= \gamma_3(a_{3m+k}) = \gamma_3(b_{7n+k}) = \gamma_3(b_{9n+k}) = \gamma_3(b_{9n+m+k})\\ &= \gamma_3(b_{15n+m+k}) = \gamma_3(b_{21n+m+k}) = \gamma_3(b_{25n+m+k})\\ &= \gamma_3(b_{9n+2m+k}) = \gamma_3(b_{25n+2m+k}) = \gamma_3(b_{15n+3m+k})\\ &= \gamma_3(b_{21n+3m+k}) = \omega \quad \text{for } k \leqslant m.\end{aligned}$$

Therefore, $(J_3, \gamma_3) \sim (J_4, \gamma_3)$ with $J_4 = J_3 \cup \lambda_m(G) \cup D_m(1, r_1 + r_2)$, by 3.7 and 3.4. But $(J_4, \gamma_3) \ll (D_m(1, r_1 + r_2) \cup \lambda_m(G), \alpha^a \vee \lambda_m(\alpha)) = \delta_m(s, r_1 + r_2)$, since $\gamma_3^a = (\alpha^a \vee \lambda_m(\alpha))^a$ and $\gamma_3(b_k) = \alpha(b_k) = (\alpha^a \vee \lambda_m(\alpha))(b_k)$ for $k \leqslant 2m$. The required inclusion then follows from 3.10, completing the proof of 3.16.

We now give a formula for addition in abelian categories constructed from abelian lattices. We can then define ζ_A and complete the verification of 2.6. We will use the notations **2**, **T**, A^1/A^0, $S(A, B)$, $SI(A, B)$, f^-, $g \circ f$, f^{-1} and the terms "isorepresentative", "left sequence" and "mixed sequence" taken from [7]. Also, the notations σ and τ and the results established in [7, 3.19, p. 173] will be used without reference.

3.17. *Suppose L is an abelian lattice, A, B, C, D is a left sequence of maps $\mathbf{2} \to L$, suppose $E = E^1/E^0$ for $E^i = C^i \vee D^i$, and let g be in $SI(C, A)$ and h be in $SI(D, A)$. For any f_1, f_2 in $S(A, B)$, $\tau(f_1) + \tau(f_2) = \tau(d \circ c)$ in $\mathbf{A}_L$, where c in $S(A, E)$ satisfies $c^- = (g^- \vee D^1) \wedge (h^- \vee C^1)$ and d in $S(E, B)$ satisfies $d^- = (f_1 \circ g)^- \vee (f_2 \circ h)^-$.*

PROOF. We show first that $f \in S(X, Y)$ and $f_0 \in S(Y, X)$ such that $f^- \subset f_0^-$ implies $\tau(f_0)\tau(f) = 1_X$ in $\mathbf{A}_L$. By [7, 3.11, p. 168], choose g_0 in $SI(X, Z)$ for some Z such that X, Y, Z is a left sequence. Then $(g_0 \circ f_0) \circ f$ is defined, and $f^- \subset f_0^-$ and $f^- \vee Y^1 = X^1 \vee Y^1$ imply that $g_0^- \subset (g_0^- \vee f_0^-) \wedge (X^1 \vee Y^1 \vee Z^1) = f^- \vee (g_0 \circ f_0)^-$, using modularity. But then $g_0^- \subset ((g_0 \circ f_0) \circ f)^-$, and so $g_0 = (g_0 \circ f_0) \circ f$ by [7, 3.4]. Then

$$\tau(g_0) = \tau(g_0)\tau(f_0)\tau(f),$$

and the desired result $\tau(f_0)\tau(f) = 1_X$ follows because $\tau(g_0)$ is an isomorphism.

Now assume the hypotheses of 3.17. By [7, 3.23, p. 176], there exist u_1, u_2 in $S(A, E)$ and p_1, p_2 in $S(E, A)$ such that $u_1^- = g^- \vee D^0$, $u_2^- = h^- \vee C^0$, $p_1^- = g^- \vee D^1$ and $p_2^- = h^- \vee C^1$. Since $u_1^- \subset p_1^-$ and $u_2^- \subset p_2^-$, we have $\tau(p_1)\tau(u_1) = \tau(p_2)\tau(u_2) = 1_A$ in $\mathbf{A}_L$. By modularity and [7, 3.2, 3.3, p. 163], we have $I(u_1) = C^1 \vee D^0 = K(p_2)$ and $I(u_2) = C^0 \vee D^1 = K(p_1)$, and so $(\tau(u_1), \tau(p_2))$ and $(\tau(u_2), \tau(p_1))$ are exact in $\mathbf{A}_L$ by [7, 3.25, p. 178]. Therefore, $E = A \oplus A$ and $\tau(u_1)$, $\tau(u_2)$, $\tau(p_1)$ and $\tau(p_2)$ form a direct sum system [5, Theorem 2.42, p. 51]. Using [7, 3.3] and modularity, we can verify that c in $S(A, E)$ and d in $S(E, B)$ exist with c^- and d^- as given above. Now $c^- = p_1^- \wedge p_2^-$, so $\tau(p_1)\tau(c) = \tau(p_2)\tau(c) = 1_A$ by our previous result. Therefore, $\tau(c) = (1_A, 1_A)\colon A \to A \oplus A$ in $\mathbf{A}_L$. Furthermore, $f_1 = (f_1 \circ g) \circ g^{-1}$, and so

$$f_1^- = (g^- \vee (f_1 \circ g)^-) \wedge (A^1 \vee B^1) \subset (u_1^- \vee d^-) \wedge (A^1 \vee B^1) = (d \circ u_1)^-.$$

Therefore, $f_1 = d \circ u_1$ by [7, 3.4], and so $\tau(f_1) = \tau(d)\tau(u_1)$. Since $\tau(f_2) = \tau(d)\tau(u_2)$ similarly, we have the equation below, in $\mathbf{A}_L$:

$$\tau(d) = \begin{pmatrix}\tau(f_1)\\ \tau(f_2)\end{pmatrix}\colon A \oplus A \to B.$$

Now A, B, E is a left sequence by [7, 3.1, 3.2, p. 163], so $d \circ c$ is defined. But then $\tau(f_1) + \tau(f_2) = \tau(d)\tau(c) = \tau(d \circ c)$, using the (left) definition of sum in an abelian category [5, p. 47]. This completes the proof of 3.17.

DEFINITION. Suppose $r \in R$, A is in $\mathbf{A}_{M(K;R)}$ (that is, $A\colon 2 \to M(K; R)$) and $m \geq |A^1|$. Let $\pi_m(A)$ denote $\pi_m(A^1)/\pi_m(A^0)$ in $\mathbf{A}_{M(K;R)}$, since $\pi_m(A^0) \subset \pi_m(A^1)$ by 3.13. Let $\zeta_A^m(r)$ denote the map $f\colon \mathbf{T} \to M(K; R)$ in $S(\pi_m(A), A)$ such that $f^- = A^0 \vee \delta_m(A^1, r)$. (By 3.13 and [7, 3.4, p. 164], such an f exists uniquely.) Again by 3.13, $\zeta_A^m(1)$ is isorepresentative, belonging to $SI(\pi_m(A), A)$. Using [7, 3.19], define $\zeta_A\colon R \to \mathrm{Hom}(A, A)$ by setting $\zeta_A(r)$ equal to $\tau\zeta_A^n(r)\tau\zeta_A^n(1)^{-1}\colon A \to A$ for $n = |A^1|$.

3.18. *For every A in $\mathbf{A}_{M(K;R)}$, $\zeta_A\colon R \to \mathrm{Hom}(A, A)$ is a ring homomorphism preserving the unit. If* $\mathbf{f}\colon A \to B$ *in* $\mathbf{A}_{M(K;R)}$ *and $r \in R$, then $\zeta_B(r)\mathbf{f} = \mathbf{f}\zeta_A(r)$.*

PROOF. We first establish the following lemma: If r_1, r_2 are in R, A, B are in $\mathbf{A}_{M(K;R)}$ and $\mathbf{g}\colon A \to \pi_m(B)$ for $m \geq |A^1|, |B^1|$ and $p \geq 2m$, then $\tau\zeta_B^m(r_1)\mathbf{g}\tau\zeta_A^p(r_2) = \tau\zeta_B^m(1)\mathbf{g}\tau\zeta_A^p(r_1 r_2)$. Assuming the lemma hypotheses, we note that $A, \pi_m(B), \pi_p(A)$ and $B, \pi_m(B), \pi_p(A)$ are left sequences in $M(K; R)$ by 3.13. Therefore, $f_1 = \zeta_B^m(r_1) \circ (\sigma(\mathbf{g}) \circ \zeta_A^p(r_2))$ and $f_2 = \zeta_B^m(1) \circ (\sigma(\mathbf{g}) \circ \zeta_A^p(r_1 r_2))$ both exist in $S(\pi_p(A), B)$. We now apply 3.15 with $x = A^1$, $y = B^1$ and $z = \sigma(\mathbf{g})^-$. Defining $w(r)$ as in 3.15 and observing that $A^0 \subset \sigma(\mathbf{g})^-$

$= z$, we find that $w(r) = (\sigma(g) \circ \zeta_A^p(r))^-$. But then

$$f_1^- = (w(r_2) \vee B^0 \vee \delta_m(B^1, r_1)) \wedge (\pi_p(A^1) \vee B^1)$$
$$\subset (w(r_1 r_2) \vee B^0 \vee \delta_m(B^1, 1)) \wedge (\pi_p(A^1) \vee B^1) = f_2^-,$$

using modularity and 3.15. So, $f_1 = f_2$ by [7, 3.4], and $\tau\zeta_B^m(r_1)g\tau\zeta_A^p(r_2) = \tau f_1 = \tau f_2 = \tau\zeta_B^m(1)g\tau\zeta_A^p(r_1 r_2)$ follows, completing the proof of the lemma.

Assume the hypotheses of 3.18, suppose $r_1, r_2 \in R$, and let $n = |A^1|$. We will compute $\tau\zeta_A^n(r_1) + \tau\zeta_A^n(r_2)$ using 3.17. Now, $\pi_n(A), A, \pi_{2n}(A), \pi_{3n}(A)$ is a left sequence in $M(K; R)$ by 3.13, and we have $g = \zeta_A^n(1)^{-1} \circ \zeta_A^{2n}(1)$ in $SI(\pi_{2n}(A), \pi_n(A))$ and $h = \zeta_A^n(1)^{-1} \circ \zeta_A^{3n}(1)$ in $SI(\pi_{3n}(A), \pi_n(A))$. Letting $E^j = \pi_{2n}(A^j) \vee \pi_{3n}(A^j)$ for $j = 0, 1$, we have by 3.17 that $\sigma(\tau\zeta_A^n(r_1) + \tau\zeta_A^n(r_2)) = d \circ c$, where c in $S(\pi_n(A), E)$ satisfies $c^- = (g^- \vee \pi_{3n}(A^1)) \wedge (h^- \vee \pi_{2n}(A^1))$ and d in $S(E, A)$ satisfies $d^- = (\zeta_A^n(r_1) \circ g)^- \vee (\zeta_A^n(r_2) \circ h)^-$. Now, $\tau(\zeta_A^n(r_1) \circ g) = \tau\zeta_A^n(r_1)\tau\zeta_A^n(1)^{-1}\tau\zeta_A^{2n}(1) = \tau\zeta_A^n(1)\tau\zeta_A^n(1)^{-1}\tau\zeta_A^{2n}(r_1) = \tau\zeta_A^{2n}(r_1)$ by the lemma, and so $\zeta_A^n(r_1) \circ g = \zeta_A^{2n}(r_1)$. Similarly, $\zeta_A^n(r_2) \circ h = \zeta_A^{3n}(r_2)$, and so $d^- = A^0 \vee \delta_{2n}(A^1, r_1) \vee \delta_{3n}(A^1, r_2)$. Define y_1, y_2 and y_3 as in 3.16, with A^1 replacing x. Now, $y_1 \vee \pi_{2n}(A^0) = g^-$, using $\pi_{2n}(A^0) \vee \delta_{2n}(A^1, 1) = A^0 \vee \delta_{2n}(A^1, 1)$ from 3.13, and modularity. Similarly, $y_2 \vee \pi_{3n}(A^0) = h^-$, and so $c^- = y_3 \vee \pi_{2n}(A^0) \vee \pi_{3n}(A^0)$ by 3.13 and modularity again. But $\pi_{2n}(A^0) \vee \pi_{3n}(A^0) = E^0 \subset d^-$, since d is in $S(E, A)$. Therefore, using modularity and 3.16, we have

$$\sigma(\tau\zeta_A^n(r_1) + \tau\zeta_A^n(r_2))^- = (d \circ c)^- = (y_3 \vee d^-) \wedge (\pi_n(A^1) \vee A^1)$$
$$= A^0 \vee [(y_3 \vee \delta_{2n}(A^1, r_1) \vee \delta_{3n}(A^1, r_2)) \wedge (\pi_n(A^1) \vee A^1)]$$
$$\subset A^0 \vee \delta_n(A^1, r_1 + r_2) = \zeta_A^n(r_1 + r_2)^-.$$

But then $\tau\zeta_A^n(r_1 + r_2) = \tau\zeta_A^n(r_1) + \tau\zeta_A(r_2)$ by [7, 3.4], and so

$$\zeta_A(r_1 + r_2) = \tau\zeta_A^n(r_1 + r_2)\tau\zeta_A^n(1)^{-1} = (\tau\zeta_A^n(r_1) + \tau\zeta_A^n(r_2))\tau\zeta_A^n(1)^{-1}$$
$$= \tau\zeta_A^n(r_1)\tau\zeta_A^n(1)^{-1} + \tau\zeta_A^n(r_2)\tau\zeta_A^n(1)^{-1} = \zeta_A(r_1) + \zeta_A(r_2),$$

using distributivity of composition over sum in abelian categories [5, 2.37, p. 48].

If $m \geqslant |A^1|$ and $p \geqslant 2m$, we have

$$\tau\zeta_A^m(r)\tau\zeta_A^m(1)^{-1} = \tau\zeta_A^m(r)\tau\zeta_A^m(1)^{-1}\tau\zeta_A^p(1)\tau\zeta_A^p(1)^{-1}$$
$$= \tau\zeta_A^m(1)\tau\zeta_A^m(1)^{-1}\tau\zeta_A^p(r)\tau\zeta_A^p(1)^{-1} = \tau\zeta_A^p(r)\tau\zeta_A^p(1)^{-1},$$

using the lemma. So, $\zeta_A(r) = \tau\zeta_A^m(r)\tau\zeta_A^m(1)^{-1}$ for all $m \geqslant |A^1|$.

If $m \geqslant 2n$, then

$$\zeta_A(r_1r_2) = \tau\zeta_A^n(1)\tau\zeta_A^n(1)^{-1}\tau\zeta_A^m(r_1r_2)\tau\zeta_A^m(1)^{-1}$$
$$= \tau\zeta_A^n(r_1)\tau\zeta_A^n(1)^{-1}\tau\zeta_A^m(r_2)\tau\zeta_A^m(1)^{-1} = \zeta_A(r_1)\zeta_A(r_2),$$

using the lemma and the result above. Since $\zeta_A(1) = \tau\zeta_A^n(1)\tau\zeta_A^n(1)^{-1} = 1_A$, we have proved that ζ_A is a ring homomorphism preserving the unit.

Suppose f: $A \to B$ in $\mathrm{A}_{M(K;R)}$ and $r \in R$. Choosing $m \geqslant |A^1|$, $|B^1|$ and $p \geqslant 2m$, we see that

$$\zeta_B(r)\mathbf{f} = \tau\zeta_B^m(r)\tau\zeta_B^m(1)^{-1}\mathbf{f}\tau\zeta_A^p(1)\tau\zeta_A^p(1)^{-1}$$
$$= \tau\zeta_B^m(1)\tau\zeta_B^m(1)^{-1}\mathbf{f}\tau\zeta_A^p(r)\tau\zeta_A^p(1)^{-1} = \mathbf{f}\zeta_A(r),$$

applying the lemma with $\mathrm{g} = \tau\zeta_B^m(1)^{-1}\mathbf{f}$. This completes the proof of 3.18.

Since 2.6 follows from 3.2, 3.12, 3.14, and 3.18, we have proved the main theorem. In conclusion, some additional properties of the $M(K; R)$ construction are noted.

Let Lat_ω denote the category of lattices with smallest element ω and homomorphisms preserving ω, and let CRng_1 denote the category of commutative rings with unit and unit-preserving ring homomorphisms. Since the construction of $M(K; R)$ did not use our previous assumption that K has a largest element, we can obtain a bifunctor $\mathrm{Lat}_\omega \times \mathrm{CRng}_1 \to \mathrm{Lat}_\omega$ by defining $M(f; h)$: $M(K; R) \to M(L; S)$ for f: $K \to L$ in Lat_ω and h: $R \to S$ in CRng_1 as follows:

$$M(f; h)(x) = [\{hg\colon g \in G\}, f\alpha] \quad \text{for } x = [G, \alpha].$$

(Recall that g in $F_R(V)$ is a coefficient function $V \to R$ such that $g(v) = 0$ for all but finitely many elements v in V.) Then the following proposition can be verified directly.

3.19. *Given f: $K \to L$ in Lat_ω and h: $R \to S$ in CRng_1, the expression for $M(f; h)(x)$ determines a well-defined homomorphism $M(f; h)$: $M(K; R) \to M(L; S)$, and $M(K; R)$ and $M(f; h)$ determine a covariant bifunctor M: $\mathrm{Lat}_\omega \times \mathrm{CRng}_1 \to \mathrm{Lat}_\omega$. If f and h are onto, then $M(f; h)$ is onto. If P_1: $\mathrm{Lat}_\omega \times \mathrm{CRng}_1 \to \mathrm{Lat}_\omega$ is the projection functor ($P_1(K, R) = K$ and $P_1(f, h) = f$), then $\psi(K, R)$: $K \to M(K; R)$ determines a natural transformation ψ: $P_1 \to M$.*

Since $L(R)$ is a quasivariety, it is a reflective subcategory of the category of all meet and join algebras and their homomorphisms. (See [16, pp. 88–90] and [19] for discussions of reflective subcategories and their relationship to quasivarieties of universal algebras; note that "reflection" here and in [16] is "coreflection" in [19].) In particular, every K in Lat_ω has a smallest congruence q such that K/q is in $L(R)$. The universal property for reflection implies the following: Every $K \to L$ in Lat_ω such that L is in $L(R)$ can be uniquely factored $K \to K/q \to L$

in Lat_ω, where the first factor is the canonical epimorphism. The reflection map $K \to K/q$ is closely related to ψ.

3.20. *For K in* Lat_ω *and R in* CRng_1, *the reflection map h: $K \to K/q$ is the coimage of $\psi(K, R)$: $K \to M(K; R)$. That is, $\psi(K, R)$ factors uniquely as $K \xrightarrow{h} K/q \xrightarrow{f} M(K; R)$ such that f is one-one.*

PROOF. Since there is an exact embedding functor G: $\mathrm{A}_{M(K;R)} \to R$-Mod by 2.6 and 2.8, every interval sublattice of $M(K; R)$ is in $L(R)$ by [7, 3.24, p. 178]. But then $M(K; R)$ is in $L(R)$ because $L(R)$ is a quasivariety, since any universal Horn formula satisfied in every interval sublattice of $M(K; R)$ is satisfied in $M(K; R)$.

Assuming the hypotheses, we obtain the following diagram in Lat_ω:

$$\begin{array}{ccc} K & \xrightarrow{\ \psi(K,R)\ } & M(K;R) \\ {\scriptstyle h}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle M(h;\,1_R)} \\ K/q & \xrightarrow[\ \psi(K/q,R)\ ]{} & M(K/q;R) \end{array}$$

The outer rectangle is commutative because ψ is natural, and a unique f exists making the upper triangle commutative by the universal property of h for reflection. But then the lower triangle is commutative because h is onto. Furthermore, $\psi(K/q, R)$ is one-one by 2.5 and 3.2, since K/q is in $L(R)$. Therefore, f is one-one, completing the proof.

ADDED IN PROOF. Some of the results of [6] were published in: C. Herrmann and W. Poguntke, *The class of sublattices of normal subgroup lattices is not elementary,* Algebra Universalis 4 (1975), 280–286.

REFERENCES

1. E. Artin, *Geometric algebra,* Interscience, New York and London, 1957. MR 18, 553.

2. R. Baer, *Linear algebra and projective geometry,* Academic Press, New York, 1952. MR 14, 675.

3. G. Birkhoff, *Lattice theory,* 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R. I., 1967. MR 37 #2638.

4. W. Craig, *On axiomatizability within a system,* J. Symbolic Logic 18 (1953), 30–32. MR 14, 1051.

5. P. J. Freyd, *Abelian categories. An introduction to the theory of functors,* Harper's Ser. in Modern Math., Harper & Row, New York, 1964. MR 29 #3517.

6. C. Herrmann and W. Poguntke, *Axiomatic classes of lattices of normal subgroups,* Technische Hochschule Darmstadt Preprint, no. 12, Darmstadt, West Germany, 1972.

7. G. Hutchinson, *Modular lattices and abelian categories,* J. Algebra 19 (1971), 156–184. MR 43 #4880.

8. G. Hutchinson, *The representation of lattices by modules,* Bull. Amer. Math. Soc. 79 (1973), 172–176. MR 47 #1693.

9. ———, *On classes of lattices representable by modules,* Proc. University of Houston Lattice Theory Conference, Houston, Texas, 1973, pp. 69–94.

10. B. Jónsson, *On the representation of lattices,* Math. Scand. 1 (1953), 193–206. MR 15, 389.

11. ———, *Modular lattices and Desargues' theorem,* Math. Scand. 2 (1954), 295–314. MR 16, 787.

12. ———, *Representation of modular lattices and of relation algebras,* Trans. Amer. Math. Soc. 92 (1959), 449–464. MR 21 #7175.

13. S. R. Kogalovskiĭ, *Structural characteristics of universal classes,* Sibirsk. Mat Ž. 4 (1963), 97–119. (Russian) MR 26 #4909.

14. J. Łos, *Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres,* Mathematical Interpretation of Formal Systems, North-Holland, Amsterdam, 1955, pp. 98–113. MR 17, 700.

15. S. Lubkin, *Imbedding of abelian categories,* Trans. Amer. Math. Soc. 97 (1960), 410–417. MR 30 #133.

16. S. Mac Lane, *Categories for the working mathematician,* Springer-Verlag, New York and Berlin, 1971.

17. M. Makkai and G. McNulty, *Universal Horn axiom systems for lattices of submodules* (submitted).

18. B. M. Šeĭn, *Relation algebras and function semigroups,* Semigroup Forum 1 (1970), no. 1, 1–62. MR 44 #2856.

19. A. Shafaat, *On implicationally defined classes of algebras,* J. London Math. Soc. 44 (1969), 137–140. MR 38 #5691.

20. J. von Neumann, *Continuous geometry,* Princeton Math. Ser., no. 25, Princeton Univ. Press, Princeton, N. J., 1960. MR 22 #10931.

21. R. Wille, *Primitive Länge und primitive Weite bei modularen Verbänden,* Math. Z. 108 (1969), 129–136.

DIVISION OF COMPUTER RESEARCH AND TECHNOLOGY, NATIONAL INSTITUTES OF HEALTH, DEPARTMENT OF HEALTH, EDUCATION AND WELFARE (LAS), BETHESDA, MARYLAND 20014